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引入地形因素的海洋动力学方程组整体弱解的稳定性

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摘　要:研究了引入地形因素和非定常外源强迫的海洋动力学方程组.该方程组是由速度方程、温度方程和盐度方程耦合而成的.在假设初值具有一定正则性时,使用能量估计方法,证明了给出新盐度边界条件的海洋动力学方程组初边值问题整体弱解的 L^1 稳定性和几乎处处稳定性等结论.

关键词:海洋动力学方程组;整体弱解;地形因素;新盐度边界条件;稳定性

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本文研究了引入地形因素和非定常外源强迫的海洋动力学方程组,并证明了整体弱解的 L^1 稳定性和几乎处处稳定性等结论.

首先,给出参考文献[1-2]中海洋动力学方程组的具体形式.引入球坐标系 (θ,λ,z,t),其中 $\theta\in[0,\pi]$ 是余纬,$\lambda\in[0,2\pi]$ 是经度,$z\in[-\tilde{h},z_{so}]$ 是高度,$z_{so}(\theta,\lambda,t)$ 和 $-\tilde{h}(\theta,\lambda)$ 分别是海洋表面和海洋底部的海拔高度,t 是时间. 再给出海水密度、海水温度和盐度的状态方程:

$$\rho=\rho_0(1-\alpha_T(T-T_0)+\alpha_S(S-S_0)),\tag{1}$$

其中 α_T 和 α_S 是两个正常数,T_0 是某个定常的温度,S_0 是某个定常的盐度,并且当 $T=T_0$ 和 $S=S_0$ 时,有 $\rho=\rho_0$,其中 ρ_0 是常数.

接下来引入参考标准温度 $\tilde{T}(z)$,参考标准盐度 $\tilde{S}(z)$,参考标准密度 $\tilde{\rho}(z)$ 及海洋表面的海拔高度 $\tilde{z}_{so}$,并假设满足如下条件:

$$\begin{cases}\tilde{\rho}(z)=\rho_0(1-\alpha_T(\tilde{T}(z)-T_0)+\alpha_S(\tilde{S}(z)-S_0)),\\ \dfrac{\mathrm{d}\tilde{p}}{\mathrm{d}z}=-\tilde{\rho}g,\tilde{p}\mid_{z=z_{so}}=\tilde{p}_{sa},\tilde{z}_{so}=0.\end{cases}\tag{2}$$

然后再引入一个地形坐标系 (θ,λ,ζ,t),其中 $\zeta=(z-z_{so})/(z_{so}+\tilde{h})\in[-1,0]$,则系统的区域为:$O\times[0,M]:=O_S\times[-1,0]\times[0,M]=[0,\pi]\times[0,2\pi]\times[-1,0]\times[0,M]$,$M>0$.

参考标准温度 $\tilde{T}$,参考标准盐度 $\tilde{S}$,参考标准密度 $\tilde{\rho}$ 和参考标准压力 $\tilde{p}$ 也可以用 ζ 来表示,那么海水温度 $T(\theta,\lambda,\zeta,t)$ 等于 $\tilde{T}(\zeta)+T'(\theta,\lambda,\zeta,t)$,海水盐度 $S(\theta,\lambda,\zeta,t)$ 等于 $\tilde{S}(\zeta)+S'(\theta,\lambda,\zeta,t)$,海水密度 $\rho(\theta,\lambda,\zeta,t)$ 等于 $\tilde{\rho}(\zeta)+\rho'(\theta,\lambda,\zeta,t)$,压力 $p(\theta,\lambda,\zeta,t)$ 等于 $\tilde{p}(\zeta)+p'(\theta,\lambda,\zeta,t)$,海表的海拔高度 $z_{so}(\theta,\lambda,t)$ 等于 $\tilde{z}_{so}+z'_{so}(\theta,\lambda,t)$,那么海洋动力学方程组的未知量就是海水的水平速度 $V=(\nu_\theta,\nu_\lambda)$,垂直速度 $\dot{\zeta}$,温度偏差 T',盐度偏差 S',密度偏差 ρ',压力偏差 p' 和海表的高度偏差 z'_{so}.

这些未知量满足如下的海洋动力学方程组:

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$$\begin{cases}\dfrac{\partial V}{\partial t}+(V^*\cdot\nabla)V+\dot{\zeta}^*\dfrac{\partial V}{\partial\zeta}+\left(2\omega\cos\theta+\dfrac{\cot\theta}{a}\nu_\lambda\right)\begin{pmatrix}0&-1\\1&0\end{pmatrix}V+\dfrac{1}{\rho_0}\nabla p'+\\ \dfrac{g\rho'}{\rho_0}((1+\zeta)\nabla z'_{so}+\zeta\nabla\tilde{h})=\dfrac{1}{h^*}\nabla\cdot(\tilde{h}k_{hof}\nabla V)+\dfrac{\partial}{\partial\zeta}\left(\dfrac{k_{zof}}{\tilde{h}^2}\dfrac{\partial V}{\partial\zeta}\right),\\ c_{0T}\left(\dfrac{\partial T'}{\partial t}+(V^*\cdot\nabla)T'+\dot{\zeta}^*\dfrac{\partial T'}{\partial\zeta}\right)+c_{0T}c_T^2(1+\zeta)\left(\kappa_0\dfrac{\partial z'_{so}}{\partial t}+V\cdot\nabla z'_{so}-\kappa_0k_{so}\Delta z'_{so}\right)+\\ c_{0T}c_T^2(\zeta V\cdot\nabla\tilde{h}+h^*\dot{\zeta})=\dfrac{1}{h^*}\nabla\cdot(\tilde{h}k_{hof}\nabla T')+\dfrac{\partial}{\partial\zeta}\left(\dfrac{k_{zof}}{\tilde{h}^2}\dfrac{\partial T'}{\partial\zeta}\right)+\Psi,\\ \dfrac{\partial S'}{\partial t}+(V^*\cdot\nabla)S'+\dot{\zeta}^*\dfrac{\partial S'}{\partial\zeta}-c_S^2(1+\zeta)\left(\kappa_0\dfrac{\partial z'_{so}}{\partial t}+V\cdot\nabla z'_{so}-\kappa_0k_{so}\Delta z'_{so}\right)-\\ c_S^2(\zeta V\cdot\nabla\tilde{h}+h^*\dot{\zeta})=\dfrac{1}{h^*}\nabla\cdot(\tilde{h}k_{hof}\nabla S')+\dfrac{\partial}{\partial\zeta}\left(\dfrac{k_{zof}}{\tilde{h}^2}\dfrac{\partial S'}{\partial\zeta}\right),\\ \dfrac{\partial p'}{\partial\zeta}=-h^*g\rho',\\ \kappa_0\dfrac{\partial z'_{so}}{\partial t}+\nabla\cdot(h^*V)+\dfrac{\partial h^*\dot{\zeta}}{\partial\zeta}=\kappa_0k_{so}\Delta z'_{so}.\end{cases}\tag{3}$$

其中的参数请参考文献[1－2]. $\widetilde{T}(\zeta)$ 满足 $\widetilde{T}(\zeta)\in W^{1,\infty}(0,1)$,$\widetilde{T}(\zeta)$ 非负,单调递增且 $\widetilde{T}(\zeta)=O(\zeta)$.$\tilde{h}(\theta,\lambda)$ 和 $h^*(\theta,\lambda,t)$ 都是地形函数的可允许替代函数,且满足 $\tilde{h}(\theta,\lambda),\tilde{h}^{-1}(\theta,\lambda)\in W^{1,\infty}([0,\pi]\times[0,2\pi])$,$h^*$,$h^{*-1}\in W^{1,2}(0,T;W^{1,\infty}(O_S))$.$(V^*,\dot{\zeta}^*)$ 是修正的速度场.可以采用文献[2－3]中的方法得到$(V^*,\dot{\zeta}^*)$.即令$\bar{V}:=\int_{-1}^{0}V(\theta,\lambda,\zeta,t)\mathrm{d}\zeta$,然后将 $h^*\bar{V}$ 分解成有辐散部分和无辐散部分,即

$$h^*\bar{V}=\nabla(\chi-\Phi)+\nabla\Phi+\begin{pmatrix}0&-1\\1&0\end{pmatrix}\nabla\psi,\tag{4}$$

这里 $\Delta\Phi=-\kappa_0^*\partial h^*/\partial t$,$\kappa_0^*$ 为 0 或 1.同时,取 $V^*=V-(h^*)^{-1}(\chi-\Phi)$,$\dot{\zeta}^*=-(h^*)^{-1}(\int_{-1}^{\zeta}\nabla\cdot(h^*V^*)\mathrm{d}s+\kappa_0^*h_t^*(1+\zeta))$,并且当 $\zeta=-1,0$ 时,$\dot{\zeta}^*=0$.

给出海洋动力学方程组(3)的边界条件如下,其中所有未知量关于 θ 都以 π 为周期,所有未知量关于 λ 都以 2π 为周期:

$$\begin{cases}\nu_{\theta\zeta}\mid_{\zeta=-1}=\nu_{\lambda\zeta}\mid_{\zeta=-1}=T'_\zeta\mid_{\zeta=-1}=S'_\zeta\mid_{\zeta=-1}=\dot{\zeta}\mid_{\zeta=-1}=0,\\ (k_{zof}V_\zeta+k_{s1}f(\mid V\mid)V)\mid_{\zeta=0}=0,(k_{zof}T'_\zeta+k_{s2}T')\mid_{\zeta=0}=0,\\ (k_{zof}S'_\zeta\mid+k_{s3}(P+R-E)S'+\alpha\mid V_{10}\mid^3S')\mid_{\zeta=0}=0,\\ \dot{\zeta}\mid_{\zeta=0}=0,p'\mid_{\zeta=0}=\kappa_0\tilde{\rho}_{so}gz'_{so}(\theta,\lambda,t).\end{cases}\tag{5}$$

其中 $V_{10}(\theta,\lambda)$ 和 $\tilde{\rho}_{so}(\theta,\lambda)$ 分别是距地表 10 m 高风速和参考标准海洋表面密度,且满足 $V_{10}(\theta,\lambda)$,$\tilde{\rho}_{so}(\theta,\lambda)\in W^{1,\infty}([0,2\pi]\times[0,\pi])$,$k_{s1},k_{s2},k_{s3},\alpha,P,R$ 和 E 都是正常数,其中 P 代表降水作用,R 代表径流作用,E 代表蒸发作用,$f(\mid V\mid)$ 是吹风系数.边界条件(5)中的新盐度边界条件是由文献[4]在 2017 年提出并进行了数值模拟研究.

目前,有关原始方程组的适定性理论研究已经取得了很大进展.曾庆存[5]给出了大气界面符合真实物理过程的边界条件并研究了几种不同的大气环流模式动力学框架解的适定性.文献[6]还最早给出了海气耦合过程的动力学框架,并提出了海气界面符合真实物理过程的边界条件.20 世纪 90 年代初,文献[7－9]证明了 Navier-Stokes 方程及温度方程耦合的原始方程组初边值问题弱解的整体存在性等.为了更合理地描述大气运动的规律,文献[1,10]在上述研究基础上对原始方程进行了改进.文献[11]还证明了改进的大气方程组弱解的整体存在性和强解的唯一性等.黄海洋和郭柏灵[12－13]、文献[14]证明了不同的大气动力学方程组存在吸引子.

此外,有关强解的适定性研究也有一些重要的理论成果,例如:文献[15]证明了具有小初始能量的原始方程组强解的整体存在性;文献[15-16]证明了具有正常初始能量的原始方程组和大气海洋耦合方程组强解的局部存在性.2007年,文献[17]采用正斜压分解技术,证明了正压速度、斜压速度和温度的正则性估计,从而证明了具有正常初始能量的原始方程组强解的整体存在性.在此基础上,文献[18-20]还证明了大尺度干大气方程组吸引子的整体存在性,以及湿大气方程组的强解和吸引子的整体存在性等结论.文献[21-23]又证明了只带部分耗散项的大气海洋原始方程组强解的存在性等结论.

为了研究改进的大气环流模式的适定性.连汝续和曾庆存[3,24]证明了不同的动力学框架整体弱解的 L^1 稳定性,还利用叶果洛夫定理证明了整体弱解的几乎处处稳定性.文献[25]还利用正斜压分解技术证明了改进的动力学框架强解的整体存在性、唯一性及吸引子的存在性等.此外连汝续等还研究了一些有关大气和海洋环流模式的适定性问题,请参考文献[2,26-27].上述研究成果中的方法亦可用到引入地形因素和非定常外源强迫的海洋动力学方程组稳定性的研究中.

1 主要结论

本节将会给出海洋动力学方程组(3)的整体弱解的定义及有关稳定性的结论.首先,将系统(3)进行简化.利用文献[2-3]中的方法,由$(3)_5$ 和边界条件 $\dot{\zeta}|_{\zeta=0}=\dot{\zeta}|_{\zeta=-1}=0$,以及初值 $z'_{so}|_{t=0}=0$,可以用水平速度 $V=(\nu_\theta,\nu_\lambda)$,温度偏差 T',盐度偏差 S' 表示出垂直速度 $\dot{\zeta}$ 和压力偏差 p'.再定义函数 $\hat{U}:=(V,T',S')^{\mathrm{T}}$ 和 $U:=(V,T',S',z'_{so})^{\mathrm{T}}=(\hat{U},z'_{so})^{\mathrm{T}}$,就可以得到系统(3)的简化形式:

$$\begin{cases}\dfrac{\partial V}{\partial t}+(V^*\cdot\nabla)V+\left(\dfrac{1}{h^*}\displaystyle\int_{-1}^{\zeta}\nabla\cdot(h^*V^*)\mathrm{d}s-\dfrac{\kappa_0^*}{h^*}\dfrac{\partial h^*}{\partial t}(1+\zeta)\right)\dfrac{\partial V}{\partial\zeta}+(2\omega\cos\theta+\\ \dfrac{\cot\theta}{a}\nu_\lambda\Big)\begin{pmatrix}0&-1\\1&0\end{pmatrix}V+\dfrac{g}{\rho_0}\nabla(\kappa_0\tilde{\rho}_{so}z'_{so})+g\nabla(h^*\displaystyle\int_{\zeta}^{0}(-\alpha_TT'+\alpha_SS')\mathrm{d}s)+\\ g(-\alpha_TT'+\alpha_SS')((1+\xi)\nabla z'_{so}+\zeta\nabla\tilde{h})=\dfrac{1}{h^*}\nabla\cdot(\tilde{h}k_{hof}\nabla V)+\dfrac{\partial}{\partial\zeta}\left(\dfrac{k_{zof}}{\tilde{h}^2}\dfrac{\partial V}{\partial\zeta}\right),\\ \dfrac{\partial T'}{\partial t}+(V^*\cdot\nabla)T'+\left(-\dfrac{1}{h^*}\displaystyle\int_{-1}^{\zeta}\nabla\cdot(h^*V^*)\mathrm{d}s-\dfrac{\kappa_0^*}{h^*}\dfrac{\partial h^*}{\partial t}(1+\zeta)\right)\dfrac{\partial T'}{\partial\zeta}+\\ c_T^2V\cdot((1+\zeta)\nabla z'_{so}+\zeta\nabla\tilde{h})-c_T^2\displaystyle\int_{-1}^{\zeta}\nabla\cdot(h^*V^*)\mathrm{d}s=\\ \dfrac{1}{c_{0T}}\left(\dfrac{1}{h^*}\nabla\cdot(\tilde{h}k_{hof}\nabla T')+\dfrac{\partial}{\partial\zeta}\left(\dfrac{k_{zof}}{\tilde{h}^2}\dfrac{\partial T'}{\partial\zeta}\right)\right)+\dfrac{\Psi}{c_{0T}},\\ \dfrac{\partial S'}{\partial t}+(V^*\cdot\nabla)S'+\left(\dfrac{1}{h^*}\displaystyle\int_{-1}^{\zeta}\nabla\cdot(h^*V^*)\mathrm{d}s-\dfrac{\kappa_0^*}{h^*}\dfrac{\partial h^*}{\partial t}(1+\zeta)\right)\dfrac{\partial S'}{\partial\zeta}-\\ c_S^2V\cdot((1+\zeta)\nabla z'_{so}+\zeta\nabla\tilde{h})+c_S^2\displaystyle\int_{-1}^{\zeta}\nabla\cdot(h^*V^*)\mathrm{d}s=\\ \dfrac{1}{h^*}\nabla\cdot(\tilde{h}k_{hof}\nabla S')+\dfrac{\partial}{\partial\zeta}\left(\dfrac{k_{zof}}{\tilde{h}^2}\dfrac{\partial S'}{\partial\zeta}\right),\\ \kappa_0\dfrac{\partial z'_{so}}{\partial t}+\nabla\cdot(h^*\bar{V})=\kappa_0k_{so}\Delta z'_{so}.\end{cases}\tag{6}$$

接下来,类似于文献[2-3]可以给出方程组的算子和弱解的定义,首先定义 $A(U):=(\hat{U},\kappa_0\nabla z'_{so})^{\mathrm{T}}$,$F:=(0,\Psi/c_{0T},0,0)^{\mathrm{T}}$,并定义 $B(U)$ 是耗散项构成的算子,$N(U)(U)$ 是其余项构成的算子,这里省略具体的表述.然后再给出弱解的定义.

定义1(弱解的定义) 对任意 $M>0$,若 $\|U\|_{L^\infty(0,M;L^2(O))}+\|U\|_{L^23(0,M;H^2(O))}$ 有界,且在分布意义下满足 $A(U)_t+N(U)(U)+B(U)=F$,即对任何的

$$\phi=(\phi_{\nu_\theta},\phi_{\nu_\lambda},\phi_{T'},\phi_{S'},\phi_{z'_{so}})=(\phi_U,\phi_{z'_{so}})\in C^\infty(0,M;C_0^\infty(O)),$$

且有 $\phi(M,\cdot)=0$ 使得

$$(A(U_0),\phi(0,\cdot))+\int_0^M(A(U),\phi_t)\mathrm{d}t-\int_0^M(a(U,\phi)+d(U,\phi)+b(U,U,\phi)-(F,\phi))\mathrm{d}t=0,$$

成立，其中 $(\cdot,\cdot)$ 表示 $L^2(O)$ 中的内积，且有

$$a(U,\phi)=\int_O\frac{\nabla\phi_V}{h^*}\cdot(\tilde{h}k_{hof}\nabla V)\mathrm{d}\sigma\mathrm{d}\zeta+\frac{1}{c_{0T}}\int_O\frac{\nabla\phi_{T'}}{h^*}\cdot(\tilde{h}k_{hof}\nabla T')\mathrm{d}\sigma\mathrm{d}\zeta+\int_O\frac{\nabla\phi_{S'}}{h^*}\cdot$$

$$(\tilde{h}k_{hof}\nabla S')\mathrm{d}\sigma\mathrm{d}\zeta+\int_O\nabla\phi_{z'_{so}}\cdot(\kappa_0k_{so}\nabla z'_{so})\mathrm{d}\sigma\mathrm{d}\zeta+\int_O\frac{k_{zof}}{\tilde{h}^2}\frac{\partial(V,C_{0T}^{-1}T',S')}{\partial\zeta}\cdot\frac{\partial\phi_{\hat{U}}}{\partial\zeta}\mathrm{d}\sigma\mathrm{d}\zeta,\tag{7}$$

$$d(U,\phi)=k_{s1}\int_{O_S}\frac{1}{\tilde{h}^2}(f(|V|)V\cdot\phi_V)|_{\zeta=0}\mathrm{d}\sigma+\frac{k_{s2}}{c_{0T}}\int_{O_S}\frac{1}{\tilde{h}^2}(T'\phi_{T'})|_{\zeta=0}\mathrm{d}\sigma+$$

$$\alpha\int_{O_S}\frac{1}{\tilde{h}^2}(|V_{10}|^3S'\phi_{S'})|_{\zeta=0}\mathrm{d}\sigma+k_{s3}\int_{O_S}\frac{1}{\tilde{h}^2}((P+R-E)S'\phi_{S'})|_{\zeta=0}\mathrm{d}\sigma,\tag{8}$$

$$b(U,U,\phi)=\int_O(V^*\cdot\nabla)\hat{U}\cdot\phi_{\hat{U}}\mathrm{d}\sigma\mathrm{d}\zeta+\int_O\left(-\frac{1}{h^*}\int_{-1}^{\zeta}\nabla\cdot(h^*V^*)\mathrm{d}s-\frac{\kappa_0^*}{h^*}\frac{\partial h^*}{\partial t}(1+\zeta)\right)\frac{\partial\hat{U}}{\partial\zeta}\cdot\phi_{\hat{U}}\mathrm{d}\sigma\mathrm{d}\zeta+$$

$$\int_O\left(2\omega\cos\theta+\frac{\cot\theta}{\alpha}\nu_\lambda\right)(\nu_\theta\phi_{\nu_\lambda}-\nu_\lambda\phi_{\nu_\theta})\mathrm{d}\sigma\mathrm{d}\zeta+\int_O g(-\alpha_TT'+\alpha_SS')((1+\zeta)\nabla z'_{so}+$$

$$\zeta\nabla\tilde{h})\cdot\phi_V\mathrm{d}\sigma\mathrm{d}\zeta+\int_O((1+\zeta)\nabla z'_{so}+\zeta\nabla\tilde{h})\cdot(c_T^2V)\phi_{T'}\mathrm{d}\sigma\mathrm{d}\zeta+\int_O((1+\zeta)\nabla z'_{so}+$$

$$\zeta\nabla\tilde{h})\cdot(-c_S^2V)\phi_{S'}\mathrm{d}\sigma\mathrm{d}\zeta+\int_O(g\nabla h^*\int_\zeta^0(-\alpha_TT'+\alpha_SS')\mathrm{d}s)\cdot\phi_V-c_T^2\int_{-1}^{\zeta}\nabla\cdot$$

$$(h^*V^*)\mathrm{d}s\phi_{T'}+c_S^2\int_{-1}^{\zeta}\nabla\cdot(h^*V^*)\mathrm{d}s\phi_{S'}]\mathrm{d}\sigma\mathrm{d}\zeta+\int_O\nabla\cdot(h^*\bar{V})\phi_{z'_{so}}\mathrm{d}\sigma\mathrm{d}\zeta,\tag{9}$$

那么 U 称为方程组(6)在 $O\times[0,M]$ 上的弱解.

然后给出整体弱解的稳定性的结论.

定理 1(整体弱解的稳定性)　对任意 $M>0$，假定 $\Psi(\theta,\lambda,\zeta,t)\in L^2(0,T;H^{-1}(O))$，以及 $f(s)\in C(\mathbf{R}^+)$，$C_1s^\alpha\le f(s)\le C_2(1+s^\alpha)$，$0\le\alpha<1$.

令 $U^m=(\nu_\theta^m,\nu_\lambda^m,S'^m,z'^m_{so})=(\hat{U}^m,z'^m_{so})$ 是海洋动力学方程组(6)的弱解序列，且给定初值序列为 $U^m|_{t=0}=(\hat{U}_m,z'^m_{so}|_{t=0})=U_0^m$，并且假定初值序列满足如下条件，$\|U_0^m-U_0\|_{L^1(O)}\to0$ 其中 $U_0\in L^2(O)$ 且 U_0^m 对于 $m\in\mathbf{N}^+$ 满足 $\|U_0^m\|_{L^2(O)}<C$，其中 $C>0$ 表示正常数，则有 $\|U^m-U\|_{L^2(O\times[0,M])}\to0$，其中 $U:=(\hat{U},z'_{so})^{\mathrm{T}}$ 是海洋动力学方程组(6) 的弱解，并且对应的初值为 $U_0=(\hat{U}_0,z'_{so0})$.

注　由文献[3]中的方法，类似可以得到，如果 $\|U_0^m-U_0\|_{L^1(O)}\to0$，那么 $\|U^m-U\|_{L^1(O\times[0,M])}\to0$，这即是整体弱解的 L^1 稳定性.同时也可以得到，如果 $U_0^m-U_0\to0$ a.e.，并且 $U_0^m,U_0\in L^2(O)$，则 $U^m-U\to0$ a.e.，这即是整体弱解的几乎处处稳定性.

2　能量估计

利用文献[2−3]中的方法，给出海洋动力学方程组(6)的弱解序列的基本能量估计.

引理 1　对任意 $M>0$，在定理 1 的假定下，海洋动力学方程组(6) 的弱解序列 U^m 满足如下的能量估计：

$$\|(V^m,T'^m,S'^m)\|^2_{L^2(O)}+\kappa_0\|z'^m_{so}\|^2_{L^2(O_S)}+C\int_0^t\|(\nabla V^m,\nabla T'^m,\nabla S'^m)\|^2_{L^2(O)}\mathrm{d}\tau+C\int_0^t\|(V^m_\zeta,T'^m_\zeta,S'^m_\zeta)\|^2_{L^2(O)}\mathrm{d}\tau+$$

$$\kappa_0C\int_0^t\|\nabla z'^m_{so}\|^2_{L^2(O)}\mathrm{d}\tau+C\int_0^t\int_{O_S}(f(|V^m|)|V^m|^2)|_{\zeta=0}\mathrm{d}\sigma\mathrm{d}\tau+C\int_0^t\int_{O_S}|T'^m|^2|_{\zeta=0}\mathrm{d}\sigma\mathrm{d}\tau+$$

$$C\int_0^t\int_{O_S}|S'^m|^2|_{\zeta=0}\mathrm{d}\sigma\mathrm{d}\tau\le C(M)(1+\int_0^t\|\Psi\|^2_{L^2(O)}\mathrm{d}\tau),t\in[0,M].\tag{10}$$

证明 将(6)式与 $\left(h^{*}V^{m},\frac{g\alpha_{T}}{c_{T}^{2}}h^{*}T^{'m},\frac{g\alpha_{S}}{c_{S}^{2}}h^{*}S^{'m},\frac{g\tilde{\rho}_{so}}{\rho_{0}}z_{so}^{'m}\right)$ 做内积并利用边界条件、Poincaré 不等式和 Young 不等式,有

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{O}h^{*}\left(|V^{m}|^{2}+\frac{g\alpha_{S}}{c_{T}^{2}}|T^{'m}|+\frac{g\alpha_{T}}{c_{S}^{2}}|S^{'m}|^{2}\right)\mathrm{d}\sigma\mathrm{d}\zeta+\frac{g\kappa_{0}}{2\rho_{0}}\frac{\mathrm{d}}{\mathrm{d}t}\int_{O_{S}}z_{so}^{'m2}\tilde{\rho}_{so}\mathrm{d}\sigma+$$
$$k_{hof}\int_{O}\tilde{h}|\nabla V^{m}|^{2}\mathrm{d}\sigma\mathrm{d}\zeta+\frac{k_{hof}g\alpha_{T}}{c_{0T}c_{T}^{2}}\int_{O}\tilde{h}|\nabla T^{'m}|^{2}\mathrm{d}\sigma\mathrm{d}\zeta+\frac{k_{hof}g\alpha_{S}}{c_{S}^{2}}\int_{O}\tilde{h}|\nabla S^{'m}|^{2}\mathrm{d}\sigma\mathrm{d}\zeta+$$
$$\frac{k_{so}g\kappa_{0}}{\rho_{0}}\int_{O_{S}}|\nabla z_{so}^{'m}|^{2}\tilde{\rho}_{so}\mathrm{d}\sigma\mathrm{d}\zeta+k_{zof}\int_{O}\frac{h^{*}}{\tilde{h}^{2}}|V_{\zeta}^{m}|^{2}\mathrm{d}\sigma\mathrm{d}\zeta+\frac{k_{zof}g\alpha_{T}}{c_{0T}c_{T}^{2}}\int_{O}\frac{h^{*}}{\tilde{h}^{2}}|T_{\zeta}^{'m}|^{2}\mathrm{d}\sigma\mathrm{d}\zeta+$$
$$\frac{k_{zof}g\alpha_{S}}{c_{S}^{2}}\int_{O}\frac{h^{*}}{\tilde{h}^{2}}|S_{\zeta}^{'m}|^{2}\mathrm{d}\sigma\mathrm{d}\zeta+k_{s1}\int_{O_{S}}\frac{h^{*}}{\tilde{h}^{2}}(f(|V^{m}|)V^{m2})|_{\zeta=0}\mathrm{d}\sigma+\frac{k_{s2}g\alpha_{T}}{c_{0T}c_{T}^{2}}\int_{O_{S}}\frac{h^{*}}{\tilde{h}^{2}}T^{'m2}|_{\zeta=0}\mathrm{d}\sigma+$$
$$\frac{g\alpha\alpha_{S}}{c_{S}^{2}}\int_{O_{S}}\frac{h^{*}}{\tilde{h}^{2}}|V_{10}|^{3}S^{'m2}|_{\zeta=0}\mathrm{d}\sigma\leq C\int_{O}h^{*}\left(|V^{m}|^{2}+\frac{g\alpha_{T}}{c_{T}^{2}}|T^{'m}|^{2}+\right.$$
$$\left.\frac{g\alpha_{S}}{c_{S}^{2}}|S^{'m}|^{2}\right)\mathrm{d}\sigma\mathrm{d}\zeta+C\kappa_{0}\int_{O_{S}}\tilde{\rho}_{so}z_{so}^{'m2}\mathrm{d}\sigma+C\|\Psi\|_{L^{2}(O)}^{2},\tag{11}$$

再对上式使用 Gronwall 不等式可以得到(10)式.

3 整体弱解稳定性的证明

由引理 1 可以得到弱解序列 U^{m} 的 $L^{\infty}(0,M;L^{2}(O))$ 和 $L^{2}(0,M;H^{1}(O))$ 范数有界. 接下来证明整体弱解的稳定性.

引理 2 令 U^{m} 是海洋动力学方程组(6)的弱解序列,则 U^{m} 存在子列且满足

$$\|U^{m}-U\|_{L^{2}(0,M;L^{2}(O))}\rightarrow 0,\tag{12}$$

以及 $\int_{0}^{M}(A(U^{m}),\phi_{t})\mathrm{d}t\rightarrow\int_{0}^{M}(A(U),\phi_{t})\mathrm{d}t$,其中 $\phi=(\phi_{\nu_{\theta}},\phi_{\nu_{\lambda}},\phi_{T'},\phi_{S'},\phi_{z'_{so}})=(\phi_{\hat{U}},\phi_{z'_{so}})\in C^{\infty}(0,M;C_{0}^{\infty}(O))$ 且有 $\phi(M,\cdot)=0$.

证明 取定试验函数 $\varphi=(\varphi_{\nu_{\theta}},\varphi_{\nu_{\lambda}},\varphi_{T'},\varphi_{S'},\varphi_{z'_{so}})\in H^{2}(O)$,则由(10)式可得如下估计

$$(B(U^{m}),\varphi)=-\int_{O}\frac{\nabla\varphi_{V}}{h^{*}}\cdot(\tilde{h}k_{hof}\nabla V^{m})\mathrm{d}\sigma\mathrm{d}\zeta-\frac{1}{c_{0T}}\int_{O}\frac{\nabla\varphi_{T}}{h^{*}}\cdot(\tilde{h}k_{hof}\nabla T^{'m})\mathrm{d}\sigma\mathrm{d}\zeta-$$
$$\int_{O}\frac{\nabla\varphi_{S'}}{h^{*}}\cdot(\tilde{h}k_{hof}\nabla S^{'m})\mathrm{d}\sigma\mathrm{d}\zeta-\int_{O}\nabla\varphi_{z'_{so}}\cdot(\kappa_{0}k_{so}\nabla z_{so}^{'m})\mathrm{d}\sigma\mathrm{d}\zeta-\int_{O}\frac{k_{zof}}{\tilde{h}^{2}}(V_{\zeta}^{m},C_{0T}^{-1}T_{\zeta}^{'m},S_{\zeta}^{'m})\cdot\frac{\partial\varphi_{\hat{U}}}{\partial\zeta}\mathrm{d}\sigma\mathrm{d}\zeta+$$
$$\int_{O_{S}}\frac{k_{zof}}{\tilde{h}^{2}}((V_{\zeta}^{m},C_{0T}^{-1}T_{\zeta}^{'m},S_{\zeta}^{'m})\cdot\varphi_{\hat{U}})|_{\zeta=0}\mathrm{d}\sigma\mathrm{d}\zeta\leq C+C(\int_{O}|\nabla_{3}U^{m}|^{2}\mathrm{d}\sigma\mathrm{d}\zeta)^{\frac{1}{2}}+(\int_{O}|\nabla_{3}U^{m}|^{2}\mathrm{d}\sigma\mathrm{d}\zeta)^{\frac{1+\alpha}{2}},\tag{13}$$

其中 $C>0$ 为正常数,则由(13)式可得 $B(U^{m})\in L^{2/(1+\alpha)}(0,M;H^{-2}(O))$.

还可以得到

$$(N(U^{m})(U^{m}),\varphi)=\int_{O}(V^{*m}\cdot\nabla)\hat{U}^{m}\cdot\varphi_{U}\mathrm{d}\sigma\mathrm{d}\zeta+\int_{O}\left(-\frac{1}{h^{*}}\int_{-1}^{\zeta}\nabla\cdot(h^{*}V^{*m})\mathrm{d}s-\frac{\kappa_{0}^{*}}{h^{*}}\frac{\partial h^{*}}{\partial t}(1+\zeta)\right)\hat{U}_{\zeta}^{m}\cdot$$
$$\varphi_{\hat{U}}\mathrm{d}\sigma\mathrm{d}\zeta+\int_{O}\left(2\omega\cos\theta+\frac{\cot\theta}{a}\nu_{\lambda}^{m}\right)(\nu_{\lambda}^{m}\varphi_{\nu_{\theta}}-\nu_{\theta}^{m}\varphi_{\nu_{\lambda}})\mathrm{d}\sigma\mathrm{d}\zeta+\int_{O}((1+\zeta)\nabla z_{so}^{'m}+\zeta\nabla\tilde{h})g(-\alpha_{T}T^{'m}+$$
$$\alpha_{S}S^{'m})\cdot\varphi_{V}\mathrm{d}\sigma\mathrm{d}\zeta+\int_{O}((1+\zeta)\nabla z_{so}^{'m}+\zeta\nabla\tilde{h})\cdot(c_{T}^{2}V^{m})\varphi_{T'}\mathrm{d}\sigma\mathrm{d}\zeta+\int_{O}((1+\zeta)\nabla z_{so}^{'m}+\zeta\nabla\tilde{h})\cdot$$
$$(-c_{S}^{2}V^{m})\varphi_{S'}\mathrm{d}\sigma\mathrm{d}\zeta+\int_{O}\left[\left(g\nabla h^{*}\int_{\zeta}^{0}(-\alpha_{T}T^{'m}+\alpha_{S}S^{'m})\mathrm{d}s\right)\cdot\varphi_{V}-c_{T}^{2}\int_{-1}^{\zeta}\nabla\cdot(h^{*}V^{m})\mathrm{d}s\varphi_{T'}+\right.$$

$$c_S^2\int_{-1}^{\zeta}\nabla\cdot(h^*V^m)\mathrm{d}S\varphi_{S'}]\mathrm{d}\sigma\mathrm{d}\zeta+\int_O\nabla\cdot(h^*\bar{V}^m)\varphi_{z'_{so}}\mathrm{d}\sigma\mathrm{d}\zeta. \tag{14}$$

再由性质 $\|V^{*m}\|_{H^1(O)}\le\|V^m\|_{H^1(O)}$ 可得如下估计成立

$$\int_O(V^{*m}\cdot\nabla)\hat{U}^m\cdot\varphi_{\hat{U}}\mathrm{d}\sigma\mathrm{d}\zeta\le C\left(\int_O|\nabla_3U^m|^2\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{1}{2}}\left(\int_O|V^{*m}|^3\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{1}{3}}\left(\int_O|\varphi|^6\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{1}{6}}\le$$
$$C\left(\int_O|\nabla_3U^m|^2\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{3}{4}}\left(\int_O|V^{*m}|^2\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{1}{4}}\left(\int_O(|\varphi|^2+|\nabla_3\varphi|^2)\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{1}{2}}\le C\left(\int_O|\nabla_3U^m|^2\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{3}{4}}, \tag{15}$$

$$\int_O\left(-\frac{1}{h^*}\int_{-1}^{\zeta}\nabla\cdot(h^*V^{*m}\mathrm{d}s-\frac{\kappa_0^*}{h^*}\frac{\partial h^*}{\partial t}(1+\zeta)\right)\hat{U}_\zeta^m\cdot\varphi_{\hat{U}}\mathrm{d}\sigma\mathrm{d}\zeta=\int_O\left(\frac{1}{h^*}\nabla\cdot(h^*V^{*m})+\right.$$
$$\left.\frac{\kappa_0^*}{h^*}\frac{\partial h^*}{\partial t}\right)\hat{U}^m\cdot\varphi_{\hat{U}}\mathrm{d}\sigma\mathrm{d}\zeta+\int_O\left(\frac{1}{h^*}\int_{-1}^{\zeta}\nabla\cdot(h^*V^{*m})\mathrm{d}s-\frac{\kappa_0^*}{h^*}\frac{\partial h^*}{\partial t}(1+\zeta)\right)\hat{U}^m\cdot\varphi_{\hat{U}_\zeta}\mathrm{d}\sigma\mathrm{d}\zeta\le$$
$$C\left(1+\int_O|\nabla\cdot V^{*m}|^2\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{1}{2}}\left(\int_O|U^m|^3\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{1}{3}}\left[\left(\int_O|\varphi_{\hat{U}}|^6\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{1}{6}}+\left(\int_O|\varphi_{\hat{U}_\zeta}|^6\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{1}{6}}\right]\le$$
$$C\left(1+\int_O|\nabla_3U^m|^2\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{3}{4}}\left(\int_O|U^m|^2\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{1}{4}}\le C\left(\int_O|\nabla_3U^m|^2\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{3}{4}}+C, \tag{16}$$

$$\int_O((1+\zeta)\nabla z_{so}'^m+\zeta\nabla\tilde{h})g(-\alpha_TT'^m+\alpha_SS'^m)\cdot\varphi_V\mathrm{d}\sigma\mathrm{d}\zeta+\int_O((1+\zeta)\nabla z_{so}'^m+$$
$$\zeta\nabla\tilde{h})\cdot c_T^2V^m\varphi_{T'}\mathrm{d}\sigma\mathrm{d}\zeta+\int_O((1+\zeta)\nabla z_{so}'^m+\zeta\nabla\tilde{h})\cdot(-c_S^2V^m)\varphi_{S'}\mathrm{d}\sigma\mathrm{d}\zeta\le$$
$$C\left(\int_O|\nabla_3U^m|^2\mathrm{d}\sigma\mathrm{d}\zeta\right)^{\frac{3}{4}}+C, \tag{17}$$

$$\int_O\left[\left(g\nabla(h^*\int_\zeta^0(-\alpha_TT'^m+\alpha_SS'^m)\mathrm{d}s\right)\cdot\varphi_V-c_T^2\int_{-1}^{\zeta}\nabla\cdot(h^*V^{*m})\mathrm{d}s\varphi_{T'}+c_S^2\int_{-1}^{\zeta}\nabla\cdot(h^*V^{*m})\mathrm{d}s\varphi_{S'}\right]\mathrm{d}\sigma\mathrm{d}\zeta\le$$
$$C\left(\left\|\left(\int_\zeta^0(-\alpha_TT'^m+\alpha_SS'^m)\mathrm{d}s\right)\right\|_{L^2(O)}+\|\nabla(-\alpha_TT'^m+\alpha_SS'^m)\|_{L^2(O)}\right)\left\|\int_{-1}^{S}\varphi_V\mathrm{d}\zeta\right\|_{L^2(O)}+C\left\|\int_{-1}^{\zeta}\nabla\cdot\right.$$
$$\left.(h^*V^{*m})\mathrm{d}s\right\|_{L^2(O)}\|\varphi_{T'}\|_{L^2(O)}+C\left\|\int_{-1}^{\zeta}\nabla\cdot(h^*V^{*m})\mathrm{d}s\right\|_{L^2(O)}\|\varphi_{S'}\|_{L^2(O)}\le C(1+\|U^m\|_{H^1(O)}), \tag{18}$$

$$\int_O\nabla\cdot(h^*\bar{V}^m)\varphi_{z'_{so}}\mathrm{d}\sigma\mathrm{d}\zeta\le C\|U^m\|_{H^1(O)}\|\varphi\|_{L^2(O)}\le C\|U^m\|_{H^1(O)}. \tag{19}$$

因此,综上可得 $N(U^m)(U^m)\in L^{\frac{3}{4}}(0,M;H^{-2}(O))$. 且有 $(F,\varphi)\le\int_O\frac{\Psi}{c_{0T}}\varphi_T\mathrm{d}\sigma\mathrm{d}\zeta\le C\|\Psi\|_{H^{-2}(O)}\|\varphi\|_{H^2(O)}\le C\|\Psi\|_{H^{-2}(O)}$.

由方程得 $U_t^m\in L^{p_t}(0,M;H^{-2}(O))$,这里的常数 $p_t=\min\{4/3,2/(1+\alpha)\}>1$.再利用 $U^m\in L^2(0,M;H^1(O))$ 和 Lions 引理可以得到 $U^m\to U\in L^2(0,M;L^2(O))$.

引理 3　令 U^m 是海洋动力学方程组(6)的弱解序列,则 U^m 存在子列且满足

$$\int_0^M a(U^m,\phi)\mathrm{d}t\to\int_0^M a(U,\phi)\mathrm{d}t, \tag{20}$$

其中 $\phi=(\phi_{\nu_\theta},\phi_{\nu_\lambda},\phi_{T'},\phi_{S'},\phi_{z'_{so}})=(\phi_{\hat{U}},\phi_{z'_{so}})\in C^\infty(0,M;C_0^\infty(O))$ 和 $\phi(M,\cdot)=0$.

证明　由 ∇_3U^m 在空间 $L^2(0,M;L^2(O))$ 中弱收敛到 ∇_3U 可得

$$\int_0^M a(U^m,\phi)\mathrm{d}t=\int_0^M\int_O\frac{\nabla\phi_V}{h^*}\cdot(\tilde{h}k_{hof}\nabla V^m)\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+\frac{1}{c_{0T}}\int_0^M\int_O\frac{\nabla\phi_{T'}}{h^*}\cdot(\tilde{h}k_{hof}\nabla T'^m)\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+$$
$$\int_0^M\int_O\frac{\nabla\phi_{S'}}{h^*}\cdot(\tilde{h}k_{hof}\nabla S'^m)\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+\int_0^M\int_O\nabla\phi_{z'_{so}}\cdot(\kappa_0k_{so}\nabla z_{so}'^m)\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+\int_0^M\int_O\frac{k_{zof}}{\tilde{h}^2}(V_\zeta^m,C_{0T}^{-1}T_\zeta'^m,S_\zeta'^m)\cdot$$
$$\phi_{\hat{U}\zeta}\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t\to\int_0^M\int_O\frac{\nabla\phi_V}{h^*}\cdot(\tilde{h}k_{hof}\nabla V)\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+\frac{1}{c_{0T}}\int_0^M\int_O\frac{\nabla\phi_{T'}}{h^*}\cdot(\tilde{h}k_{hof}\nabla T')\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+$$

$$\int_0^M\int_O \frac{\nabla\phi_{S'}}{h^*}\cdot(\tilde{h}k_{hof}\nabla S')\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+\int_0^M\int_O\nabla\phi_{z'_{so}}\cdot(\kappa_0 k_{so}\nabla z'_{so})\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+$$

$$\int_0^M\int_O\frac{k_{zof}}{\tilde{h}^2}(V_\zeta^m,C_{0T}^{-1}T_\zeta'^m,S_\zeta'^m)\cdot\phi_{\hat{U}_\zeta}\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t=\int_0^M a(U,\phi)\mathrm{d}t. \tag{21}$$

引理 4 令 U^m 是海洋动力学方程组(6)的弱解序列,则 U^m 存在子列且满足

$$\int_0^M d(U^m,\phi)\mathrm{d}t\to\int_0^M d(U,\phi)\mathrm{d}t, \tag{22}$$

其中 $\phi=(\phi_{\nu_\theta},\phi_{\nu_\lambda},\phi_{T'},\phi_{S'},\phi_{z'_{so}})=(\phi_{\hat{U}},\phi_{z'_{so}})\in C^\infty(0,M;C_0^\infty(O))$ 和 $\phi(M,\cdot)=0$.

证明 利用文献[3]中类似的方法,可以得到

$$\left|\int_0^M k_{s1}\int_{O_S}\frac{1}{\tilde{h}^2}[(f(|V^m|)V^m-f(|V|)V)\cdot\phi_V]|_{\zeta=0}\mathrm{d}\sigma\mathrm{d}t\right|\le$$

$$C\int_0^M\int_{O_S}|f(|V^m|)V^m|_{\zeta=0}-f(|V|)V|_{\zeta=0}|\mathrm{d}\sigma\mathrm{d}t\to 0, \tag{23}$$

$$\left|\int_0^M\frac{k_{s2}}{c_{0T}}\int_{O_S}\frac{1}{\tilde{h}^2}[(T'^m-T')\phi_{T'}]|_{\zeta=0}\mathrm{d}\sigma\mathrm{d}t\right|\le C(\int_0^M\int_{O_S}|T'^m-$$

$$T'|^2|_{\zeta=0}\mathrm{d}\sigma\mathrm{d}t)^{\frac{1}{2}}\left(\int_0^M\int_{O_S}|\phi_{T'}|^2|_{\zeta=0}\mathrm{d}\sigma\mathrm{d}t\right)^{\frac{1}{2}}\to 0, \tag{24}$$

$$\left|\int_0^M\int_{O_S}\frac{1}{\tilde{h}^2}[\alpha|V_{10}|^3+k_{S_3}(P+R-E)(S'^m-S')\phi_{S'}]|_{\zeta=0}\mathrm{d}\sigma\mathrm{d}t\right|\le$$

$$C\left(\int_0^M\int_{O_S}|S'^m-S'|^2|_{\zeta=0}\mathrm{d}\sigma\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^M\int_{O_S}|\phi_{S'}|^2|_{\zeta=0}\mathrm{d}\sigma\mathrm{d}t\right)^{\frac{1}{2}}\to 0. \tag{25}$$

引理 5 令 U^m 是海洋动力学方程组(6)的弱解序列,则 U^m 存在子列且满足

$$\int_0^M b(U^m,U^m,\phi)\mathrm{d}t\to\int_0^M b(U,U,\phi)\mathrm{d}t, \tag{26}$$

其中 $\phi=(\phi_{\nu_\theta},\phi_{\nu_\lambda},\phi_{T'},\phi_{S'},\phi_{z'_{so}})=(\phi_{\hat{U}},\phi_{z'_{so}})\in C^\infty(0,M;C_0^\infty(O))$ 和 $\phi(M,\cdot)=0$.

证明 可以得到

$$\int_0^M b(U^m,U^m,\phi)\mathrm{d}t=\int_0^M\int_O(V^{*m}\cdot\nabla)\hat{U}^n\cdot\phi_{\hat{U}}\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+\int_0^M\int_O\left(-\frac{1}{h^*}\int_{-1}^{\zeta}\nabla\cdot(h^*V^{*m})\mathrm{d}s-\right.$$

$$\left.\frac{\kappa_0^*}{h^*}\frac{\partial h^*}{\partial t}(1+\zeta)\right)\frac{\partial\hat{U}^m}{\partial\zeta}\cdot\phi_{\hat{U}}\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+\int_0^M\int_O\left(2\omega\cos\theta+\frac{\cot\theta}{a}\nu_\lambda^m\right)(\nu_\lambda^m\phi_{\nu_\theta}-\nu_\theta^m\phi_{\nu_\lambda})\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+$$

$$\int_O g(-\alpha_T T'^m+\alpha_S S'^m)((1+\zeta)\nabla z_{so}'^m+\zeta\nabla\tilde{h})\cdot\phi_V\mathrm{d}\sigma\mathrm{d}\zeta+\int_O((1+\zeta)\nabla z_{so}'^m+\zeta\nabla\tilde{h})\cdot$$

$$(c_T^2V^m)\phi_{T'}\mathrm{d}\sigma\mathrm{d}\zeta+\int_O((1+\zeta)\nabla z_{so}'^m+\zeta\nabla\tilde{h})\cdot(-c_S^2V^m)\phi_{S'}\mathrm{d}\sigma\mathrm{d}\zeta+$$

$$\int_0^M\int_O[(g\nabla h^*\int_\zeta^0 g(-\alpha_T T'^m+\alpha_S S'^m)\mathrm{d}s)\cdot\phi_V-c_T^2\int_{-1}^{\zeta}\nabla\cdot(h^*V^{*m})\mathrm{d}s\phi_{T'}+$$

$$c_S^2\int_{-1}^{\zeta}\nabla\cdot(h^*V^{*m})\mathrm{d}S\phi_{S'}]\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+\int_0^M\int_O\nabla\cdot(h^*\bar{V}^m)\phi_{z'_{so}}\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t, \tag{27}$$

因为在空间 $L^2(0,M;L^2(O))$ 中,$\nabla_3\cdot U^m$ 弱收敛到 $\nabla_3\cdot U$,$\int_{-1}^{\zeta}\nabla\cdot(h^*V^{*m})\mathrm{d}s$ 弱收敛到$\int_{-1}^{\zeta}\nabla\cdot(h^*V^*)\mathrm{d}s$. 所以可以证明(28)式成立

$$\int_0^M\int_O(V^{*m}\cdot\nabla)\hat{U}^m\cdot\phi_{\hat{U}}\mathrm{d}\sigma\mathrm{d}\zeta\to\int_0^M\int_O(V^*\cdot\nabla)\hat{U}\cdot\phi_{\hat{U}}\mathrm{d}\sigma\mathrm{d}\zeta. \tag{28}$$

以及

$$\int_0^M\int_O\frac{1}{h^*}\left(-\int_{-1}^{\zeta}\nabla\cdot(h^*V^{*m})\mathrm{d}S\right)\hat{U}_\zeta^m\cdot\phi_{\hat{U}}\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t=\int_0^M\int_O\frac{1}{h^*}\nabla\cdot(h^*V^{*m})\hat{U}^m\cdot\phi_{\hat{U}}\mathrm{d}\sigma\mathrm{d}\zeta\mathrm{d}t+$$

$$\int_0^M \int_O \frac{1}{h^*}\left(\int_{-1}^{\zeta} \nabla \cdot (h^* V^{*m}) \mathrm{d}S\right) \hat{U}^m \cdot \phi_{\hat{U}\zeta} \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t \to \int_0^M \int_O \frac{1}{h^*} \nabla \cdot (h^* V^*) \hat{U} \cdot \phi_{\hat{U}} \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t +$$

$$\int_0^M \int_O \frac{1}{h^*}\left(\int_{-1}^{\zeta} \nabla \cdot (h^* V^*) \mathrm{d}S\right) \hat{U} \cdot \phi_{\hat{U}\zeta} \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t = \int_0^M \int_O \frac{1}{h^*}\left(-\int_{-1}^{\zeta} \nabla \cdot (h^* V^*) \mathrm{d}s\right) \hat{U}_\zeta \cdot \phi_{\hat{U}} \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t, \quad (29)$$

$$\int_0^M \int_O \frac{1}{h^*}(-\kappa_0^* h_t^* (1+\zeta)) \hat{U}_\zeta^m \cdot \phi_{\hat{U}} \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t \to \int_0^M \int_O \frac{1}{h^*}(-\kappa_0^* h_t^* (1+\zeta)) \hat{U}_\zeta \cdot \phi_{\hat{U}} \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t, \quad (30)$$

接下来还可以证明

$$\begin{aligned}&\int_0^M [\int_O g(-\alpha_T T'^m + \alpha_S S'^m)((1+\zeta)\nabla z_{so}'^m + \zeta \nabla \tilde{h}) \cdot \phi_V \mathrm{d}\sigma \mathrm{d}\zeta + \int_O ((1+\zeta)\nabla z_{so}'^m + \zeta \nabla \tilde{h}) \cdot (c_T^2 V^m) \phi_{T'} \mathrm{d}\sigma \mathrm{d}\zeta + \\ &\int_O ((1+\zeta)\nabla z_{so}'^m + \zeta \nabla \tilde{h}) \cdot (-c_S^2 V^m) \phi_S \mathrm{d}\sigma \mathrm{d}\zeta] \mathrm{d}t \to \int_0^M [\int_O g(-\alpha_T T' + \alpha_S S')((1+ \\ &\zeta)\nabla z'_{so} + \zeta \nabla \tilde{h}) \cdot \phi_V \mathrm{d}\sigma \mathrm{d}\zeta + \int_O ((1+\zeta)\nabla z'_{so} + \zeta \nabla \tilde{h}) \cdot (c_T^2 V) \phi_{T'} \mathrm{d}\sigma \mathrm{d}\zeta + \\ &\int_O ((1+\zeta)\nabla z'_{so} + \zeta \nabla \tilde{h}) \cdot (-c_S^2 V) \phi_S \mathrm{d}\sigma \mathrm{d}\zeta] \mathrm{d}t, \quad (31)\end{aligned}$$

以及

$$\begin{aligned}&\int_0^M \int_O g \nabla h^* \int_\zeta^0 g(-\alpha_T T'^m + \alpha_S S'^m) \mathrm{d}s \cdot \phi_V \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t - \int_0^M \int_O c_T^2 \int_{-1}^{\zeta} \nabla \cdot (h^* V^m) \mathrm{d}s \phi_{T'} \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t + \\ &\int_0^M \int_O c_S^2 \int_{-1}^{\zeta} \nabla \cdot (h^* V^m) \mathrm{d}s \phi_{S'} \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t \to \int_0^M \int_O g \nabla h^* \int_\zeta^0 g(-\alpha_T T' + \alpha_S S') \mathrm{d}s \cdot \phi_V \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t - \\ &\int_0^M \int_O c_T^2 \int_{-1}^{\zeta} \nabla \cdot (h^* V) \mathrm{d}s \phi_{T'} \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t + \int_0^M \int_O c_S^2 \int_{-1}^{\zeta} \nabla \cdot (h^* V) \mathrm{d}s \phi_{S'} \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t, \quad (32)\end{aligned}$$

最后有 $\int_0^M \int_O \nabla \cdot (h^* \bar{V}^m) \phi_{z'_{so}} \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t \to \int_0^M \int_O \nabla \cdot (h^* \bar{V}) \phi_{z'_{so}} \mathrm{d}\sigma \mathrm{d}\zeta \mathrm{d}t$. 此引理证毕.

综上所述，由引理 2 至引理 5 的结论及整体弱解的定义可以证明定理 1 的结论成立.

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The stability of global weak solutions to the ocean dynamics equations with topography effects

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Abstract: This paper analyzed the ocean dynamics equations consisting of the velocity equation, the temperature equation and the salinity equation. Based on the initial data assumptions, we prove the stability of global weak solutions to the ocean dynamics equations with topography effects and non-constant external force in the case that a new ocean salinity boundary condition is given.

Keywords: the ocean dynamics equations; global weak solution; topography effects; salinity boundary condition; stability

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