

一类广义双积及其相关的 Yetter-Drinfeld 范畴

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摘要:首先给出了广义 smash 积 monoidal BiHom-代数 $(B \#_n^m H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 和广义 smash 余积 monoidal BiHom-余代数 $(B \times_q^p H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$,进而得到了由 $(B \#_n^m H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 和 $(B \times_q^p H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 构成 monoidal BiHom-双代数 $(B \#_{\times n, q}^{m, p} H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 的充分必要条件.最后构造了一类与 $(B \#_{\times n, q}^{m, p} H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 相关的张量范畴.

关键词:Radford 双积;BiHom-代数;Yetter-Drinfeld 范畴

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Radford 双积是由 D.E.Radford 在 1985 年通过 smash 积和 smash 余积构造的一类 Hopf 代数(见文献[1]),其等价于构造了一个 Yetter-Drinfeld 范畴中的 Hopf 代数,这使得该结构在有限维有点 Hopf 代数的分类中起到了关键作用.对 Radford 双积的其他研究可见文献[2-6].Hom 型代数源于对 Witt 代数和 Virasoro 代数的 q -形变的研究,近年来在 Lie 代数、Hopf 代数等相关领域得到了迅速发展,受到许多研究者的关注,见文献[4-17].特别地,文献[8]利用范畴思想给出了 Hom-型代数的推广,得到了 monoidal BiHom 型代数,这是本文研究的主要对象.文献[6]给出了 Radford 双积的一类 monoidal BiHom 型.文献[4]引入了 monoidal Hom-smash 积代数的对偶概念 monoidal Hom-smash 余积余代数,并给出了 monoidal Hom-smash 积代数和 monoidal Hom-smash 余积余代数构成 monoidal Hom-双代数的充分必要条件,并给出了相应的 Yetter-Drinfeld 模范畴,进而证明了该范畴是辫子 monoidal 范畴.文献[5]除了推广文献[4]的工作外还给出了文献[4]中 monoidal Hom-双代数的映射刻画.本文主要把文献[5]的思想利用到 monoidal BiHom 型代数中来,所得结果囊括了文献[4-6]中的主要结果.

文中所有代数系统都是定义在域 K 上的.若 C 是余代数,则用 Sweedler 记号(文献[2])表示余乘: $\Delta(c) = c_1 \otimes c_2, \forall c \in C$.“Monoidal BiHom”简记为“MBH”.

1 基本定义

本节回顾文献[8]中的一些基本概念.

定义 1 MBH 代数是一个五元组 $(A, \mu, 1_A, \alpha, \beta)$ (简记为 (A, α, β)),其中 A 是一个线性空间, $\mu: A \otimes A \rightarrow A$ 是一个线性映射(记 $\mu(a \otimes a') = aa'$), $1_A \in A, \alpha, \beta$ 是 A 上的自同态,并且对任意的 $a, a', a'' \in A$, 下面条件成立:

(A1) $\alpha \circ \beta = \beta \circ \alpha$, (A2) $\alpha(aa') = \alpha(a)\alpha(a'), \alpha(1_A) = 1_A$, (A3) $\beta(aa') = \beta(a)\beta(a'), \beta(1_A) = 1_A$, (A4) $\alpha(a)(a'a'') = (aa')\beta(a'')$, (A5) $a1_A = \alpha(a), 1_A a = \beta(a)$.

注记 1 当 $\alpha = \beta$ 时,MBH 代数是 monoidal Hom-代数.

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定义 2 MBH 余代数是一个五元组 $(C, \Delta, \varepsilon, \psi, \omega)$ (简记为 (C, ψ, ω)), 其中 C 是一个线性空间, $\Delta: C \rightarrow C \otimes C$ 是一个线性映射 (记 $\Delta(c) = c_1 \otimes c_2$), $\varepsilon: C \rightarrow K$, ψ, ω 是 C 上的自同态, 并且对任意的 $c \in C$ 下面条件成立:

$$\begin{aligned} (C1) & \psi \circ \omega = \omega \circ \psi, \\ (C2) & \psi(c)_1 \otimes \psi(c)_2 = \psi(c_1) \otimes \psi(c_2), \varepsilon \circ \psi = \varepsilon, \\ (C3) & \omega(c)_1 \otimes \omega(c)_2 = \omega(c_1) \otimes \omega(c_2), \varepsilon \circ \omega = \varepsilon, \\ (C4) & \omega(c_1) \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes \psi(c_2), \\ (C5) & \varepsilon(c_1)c_2 = \psi(c), c_1\varepsilon(c_2) = \omega(c). \end{aligned}$$

注记 2 当 $\psi = \omega$ 时, MBH 余代数是 monoidal Hom-余代数.

定义 3 MBH 双代数是一个 7 元组 $(H, \mu, 1_H, \Delta, \varepsilon, \alpha, \beta)$ (简记为 (H, α, β)), 其中 $(H, \mu, 1_H, \alpha, \beta)$ 是一个 MBH 代数, $(H, \Delta, \varepsilon, \alpha, \beta)$ 是一个 MBH 余代数且满足下面条件:

$$\begin{aligned} (D1) & \Delta(hh') = \Delta(h)\Delta(h'), \\ (D2) & \Delta(1_H) = 1_H \otimes 1_H, \\ (D3) & \varepsilon(hh') = \varepsilon(h)\varepsilon(h'), \\ (D4) & \varepsilon(1_H) = 1_H. \end{aligned}$$

此外, 如果存在线性映射 $S: H \rightarrow H$ 使得

$$\begin{aligned} (D5) & S(h_1)h_2 = h_1S(h_2) = \varepsilon(h)1_H, \\ (D6) & S \circ \alpha = \alpha \circ S, S \circ \beta = \beta \circ S, \end{aligned}$$

则称 $(H, \mu, 1_H, \Delta, \varepsilon, \alpha, \beta, S)$ (简记为 (H, α, β, S)) MBH-Hopf 代数.

定义 4 设 (A, α_A, β_A) 是 MBH 代数. 一个左 A -模是一个四元组 $(M, \triangleright, \alpha_M, \beta_M)$, 其中 M 是一个线性空间, $\triangleright: A \otimes M \rightarrow M$ 是一个线性映射, 并且 α_M, β_M 是 M 的自同态, 并且对任意的 $a, a' \in A$ 和 $m \in M$ 下面条件成立:

$$\begin{aligned} (M1) & \alpha_M \circ \beta_M = \beta_M \circ \alpha_M, \\ (M2) & \alpha_M(a \triangleright m) = \alpha_A(a) \triangleright \alpha_M(m), \\ (M3) & \beta_M(a \triangleright m) = \beta_A(a) \triangleright \beta_M(m), \\ (M4) & \alpha_A(a) \triangleright (a' \triangleright m) = (aa') \triangleright \beta_M(m), 1_A \triangleright m = \beta_M(m). \end{aligned}$$

注记 3 显然 $(A, \mu, \alpha_A, \beta_A)$ 是左 (A, α_A, β_A) -模.

定义 5 设 (C, ψ_C, ω_C) 是 MBH 余代数. 一个左 C -余模是一个四元组 $(M, \rho, \psi_M, \omega_M)$, 其中 M 是一个线性空间, $\rho: M \rightarrow C \otimes M$ (写作 $\rho(m) = m_{(-1)} \otimes m_{(0)}$) 是一个线性映射, 并且 ψ_M, ω_M 是 M 的自同态, 并且对任意的 $m \in M$ 下面条件成立:

$$\begin{aligned} (CM1) & \psi_M \circ \omega_M = \omega_M \circ \psi_M, \\ (CM2) & \psi_M(m)_{(-1)} \otimes \psi_M(m)_{(0)} = \psi_C(m_{(-1)}) \otimes \psi_M(m_{(0)}), \\ (CM3) & \omega_M(m)_{(-1)} \otimes \omega_M(m)_{(0)} = \omega_C(m_{(-1)}) \otimes \omega_M(m_{(0)}), \\ (CM4) & \omega_C(m_{(-1)}) \otimes m_{(0)(-1)} \otimes m_{(0)(0)} = m_{(-1)1} \otimes m_{(-1)2} \otimes \psi_M(m_{(0)}), \varepsilon_C(m_{(-1)})m_{(0)} = \psi_M(m). \end{aligned}$$

注记 4 显然 $(C, \Delta, \psi_C, \omega_C)$ 是左 (C, ψ_C, ω_C) -余模.

定义 6 设 (H, α_H, β_H) 是 MBH 双代数. 一个 MBH 代数 (B, α_B, β_B) 称为左 H -模 MBH 代数, 如果 (B, α_B, β_B) 是一个带有 \triangleright 作用的左 H -模, 并且对任意的 $a, b \in B, h \in H$ 下面条件成立:

$$\begin{aligned} (E1) & h \triangleright (ab) = (h_1 \triangleright a)(h_2 \triangleright b), \\ (E2) & h \triangleright 1_B = \varepsilon(h)1_B. \end{aligned}$$

定义 7 设 (H, α_H, β_H) 是 MBH 双代数. 一个 MBH 余代数 (B, α_B, β_B) 称为左 H -余模 MBH 余代数, 如果 (B, α_B, β_B) 是一个左 H -余模 (余作用为 $\rho(b) = b_{(-1)} \otimes b_{(0)}$) 并且对任意的 $b \in B$, 下面条件成立:

$$\begin{aligned} (F1) & b_{(-1)} \otimes \Delta_B(b_{(0)}) = b_{1(-1)}b_{2(-1)} \otimes b_{1(0)} \otimes b_{2(0)}, \\ (F2) & b_{(-1)}\varepsilon(b_{(0)}) = \varepsilon(b)1_H. \end{aligned}$$

定义 8 设 (H, α_H, β_H) 是 MBH 双代数. 一个 MBH 代数 (B, α_B, β_B) 称为左 H -余模 MBH 代数, 如果

(B, α_B, β_B) 是一个左 H -余模(余作用为 ρ), 并且对任意的 $a, b \in B$, 下面条件成立:

$$(G1) \rho(ab) = a_{(-1)} b_{(-1)} \otimes a_{(0)} b_{(0)},$$

$$(G2) \rho(1_B) = 1_H \otimes 1_B.$$

定义 9 设 (H, α_H, β_H) 是 MBH 双代数. 一个 MBH 余代数 (B, α_B, β_B) 称为左 H -模 MBH 余代数, 如果 (B, α_B, β_B) 是一个带有 \triangleright 作用的左 H -模, 并且对任意的 $b \in B, h \in H$ 下面条件成立:

$$(H1) \Delta(h \triangleright b) = h_1 \triangleright b_1 \otimes h_2 \triangleright b_2,$$

$$(H2) \varepsilon_B(h \triangleright b) = \varepsilon_H(h) \varepsilon_B(b).$$

2 广义双积结构

本节引入广义 smash 积 MBH 代数和广义 smash 余积 MBH 余代数, 进而给出两者成为 MBH 双代数的充要条件. 这里还给出此双代数的对极. 从现在开始, 假设所有代数结构的结构映射可逆, 比如在 MBH 代数 (H, α_H, β_H) 中要求 α_H, β_H 可逆.

命题 1 设 (H, α_H, β_H) 是 MBH 双代数, (B, α_B, β_B) 是左 H -模 MBH 代数, $m, n \in \mathbb{Z}$. 则 $(B \#_n^m H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ ($= B \otimes H$ 作为向量空间) 是 MBH 代数, 其中单位元为 $1_B 1_H$, 乘法为

$$(a \otimes h)(b \otimes g) = (a(\alpha_H^m \beta_H^n(h_1) \triangleright \beta_B^{-1}(b))) \otimes \beta_H(h_2)g,$$

其中 $a, b \in B, h, g \in H$. 在这种情况下, 称 $(B \#_n^m H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 为广义 smash 积 MBH 代数.

证明 直接证明 BiHom 结合性可得.

注记 5 (1) 当 $m=1, n=-1$ 时可得文献[6]中命题 1.

(2) 当 $\alpha=\beta, m=0, n=0$ 时可得文献[4]中定义 2.7.

(3) 当 $\alpha=\beta, n=0$ 时可得文献[5]中命题 3.1.

对偶命题 1 可得:

命题 2 设 (H, α_H, β_H) 是 MBH 双代数, (B, α_B, β_B) 是左 H -余模 MBH 余代数, 并且 $p, q \in \mathbb{Z}$. 则 $(B \times_q^p H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ ($= B \otimes H$ 作为向量空间) 是 MBH 余代数, 其中余单位为 $\varepsilon_B \otimes \varepsilon_H$, 余乘为

$$\Delta(b \otimes h) = b_1 \otimes \alpha_H^p \beta_H^q(b_{2(-1)}) \beta_H^{-1}(h_1) \otimes \beta_B(b_{2(0)}) \otimes h_2,$$

其中 $b \in B, h \in H$. 在这种情况下, 称 $(B \times_q^p H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 广义 smash 余积 MBH 余代数.

注记 6 (1) 当 $p=-1, q=1$ 时可得文献[6]中命题 2.

(2) 当 $\alpha=\beta, p=0, q=0$ 时可得文献[4]中命题 3.2.

(3) 当 $\alpha=\beta, q=0$ 时可得文献[5]中命题 3.5.

定理 1 设 (H, α_H, β_H) 是 MBH 双代数, $(B \#_n^m H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 是广义 smash 积 MBH 代数, $(B \times_q^p H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 是广义 smash 余积 MBH 余代数, 则下面条件等价: (i) $(\{B \times_{n,q}^{m,p} H, m_{B \#_n^m H}, 1_B \otimes 1_H, \Delta_{B \times_q^p H}, \varepsilon_B \otimes \varepsilon_H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H\}$ 是 MBH 双代数. (ii) 下面条件成立:

$$(R1) \varepsilon_B \text{ 是代数同态, } \Delta_B(1_B) = 1_B \otimes 1_B,$$

$$(R2) (B, \alpha_B, \beta_B) \text{ 是左 } (H, \alpha_H, \beta_H)\text{-MBH 模余代数,}$$

$$(R3) (B, \alpha_B, \beta_B) \text{ 是左 } (H, \alpha_H, \beta_H)\text{-MBH 余模代数,}$$

$$(R4) \Delta_B(ab) = a_1(\alpha_H^{m+p} \beta_H^{n+q}(a_{2(-1)}) \triangleright \beta_B^{-1}(b_1)) \otimes \beta_B(a_{2(0)}) b_2,$$

$$(R5) \alpha_H^p \beta_H^{q-1}((\alpha_H^m \beta_H^{n+1}(h_1) \triangleright b)_{(-1)}) h_2 \otimes (\alpha_H^m \beta_H^{n+1}(h_1) \triangleright b)_{(0)} = h_1 \alpha_H^{p+1} \beta_H^{q-1}(b_{(-1)}) \otimes \alpha_H^{m-1} \beta_H^{n+1}(h_2) \triangleright b_{(0)},$$

其中 $a, b \in B, h \in H, m, n, p, q \in \mathbb{Z}$. 在这种情况下, 称 $(B \times_{n,q}^{m,p} H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 广义双积 MBH 双代数.

证明 ((i) \Rightarrow (ii)) 直接验证可得.

((ii) \Rightarrow (i)) 对任意的 $a, b \in B, h, g \in H$, 有

$$\begin{aligned} \Delta_{B \times_q^p H}((a \otimes h)(b \otimes g)) &= (a(\alpha_H^m \beta_H^n(h_1) \triangleright \beta_B^{-1}(b)))_1 \otimes \alpha_H^p \beta_H^q((a(\alpha_H^m \beta_H^n(h_1) \triangleright \\ &\beta_B^{-1}(b)))_{2(-1)}) \beta_H^{-1}(\beta_H(h_{21})g_1) \otimes \beta_B((a(\alpha_H^m \beta_H^n(h_1) \triangleright \beta_B^{-1}(b)))_{2(0)}) \otimes \beta_H(h_{22})g_2 \end{aligned} \quad (R4)$$

$$\begin{aligned}
 & a_1(\alpha_H^{m+p}\beta_H^{n+q}(a_{2(-1)})) \triangleright \beta_B^{-1}((\alpha_H^m\beta_H^n(h_1) \triangleright \beta_B^{-1}(b)))_1) \otimes \alpha_H^p\beta_H^q((\beta_B(a_{2(0)})(\alpha_H^m\beta_H^n(h_1) \triangleright \\
 & \beta_B^{-1}(b)))_2)_{(-1)}(h_{21}\beta_H^{-1}(g_1)) \otimes \beta_B((\beta_B(a_{2(0)})(\alpha_H^m\beta_H^n(h_1) \triangleright \beta_B^{-1}(b)))_2)_{(0)}) \otimes \beta_H(h_{22})g_2 \quad \text{(H1)} \\
 & a_1(\alpha_H^{m+p}\beta_H^{n+q}(a_{2(-1)})) \triangleright \beta_B^{-1}(\alpha_H^m\beta_H^n(h_{11}) \triangleright \beta_B^{-1}(b_1))) \otimes \alpha_H^p\beta_H^q((\beta_B(a_{2(0)})(\alpha_H^m\beta_H^n(h_{12}) \triangleright \\
 & \beta_B^{-1}(b_2)))_{(-1)}(h_{21}\beta_H^{-1}(g_1)) \otimes \beta_B((\beta_B(a_{2(0)})(\alpha_H^m\beta_H^n(h_{12}) \triangleright \beta_B^{-1}(b_2)))_{(0)}) \otimes \beta_H(h_{22})g_2 \quad \text{(G1)} \\
 & a_1(\alpha_H^{m+p}\beta_H^{n+q}(a_{2(-1)})) \triangleright \beta_B^{-1}(\alpha_H^m\beta_H^n(h_{11}) \triangleright \beta_B^{-1}(b_1))) \otimes \alpha_H^p\beta_H^q(\beta_B(a_{2(0)})_{(-1)}(\alpha_H^m\beta_H^n(h_{12}) \triangleright \\
 & \beta_B^{-1}(b_2)))_{(-1)}(h_{21}\beta_H^{-1}(g_1)) \otimes \beta_B(\beta_B(a_{2(0)})_{(0)}(\alpha_H^m\beta_H^n(h_{12}) \triangleright \beta_B^{-1}(b_2)))_{(0)}) \otimes \beta_H(h_{22})g_2 \quad \text{(CM2)} \\
 & a_1(\alpha_H^{m+p}\beta_H^{n+q}(a_{2(-1)})) \triangleright \beta_B^{-1}(\alpha_H^m\beta_H^n(h_{11}) \triangleright \beta_B^{-1}(b_1))) \otimes \alpha_H^p\beta_H^q(\beta_B(a_{2(0)(-1)})(\alpha_H^m\beta_H^n(h_{12}) \triangleright \\
 & \beta_B^{-1}(b_2)))_{(-1)}(h_{21}\beta_H^{-1}(g_1)) \otimes \beta_B(\beta_B(a_{2(0)(0)})(\alpha_H^m\beta_H^n(h_{12}) \triangleright \beta_B^{-1}(b_2)))_{(0)}) \otimes \beta_H(h_{22})g_2 \quad \text{(C4)} \\
 & a_1(\alpha_H^{m+p}\beta_H^{n+q}(a_{2(-1)})) \triangleright \beta_B^{-1}(\alpha_H^m\beta_H^n(h_{11}) \triangleright \beta_B^{-1}(b_1))) \otimes (\alpha_H^p\beta_H^{q+1}(a_{2(0)(-1)})\alpha_H^p\beta_H^q((\alpha_H^{m+1}\beta_H^n(h_{121}) \triangleright \\
 & \beta_B^{-1}(b_2)))_{(-1)}))(\beta_H(h_{122})\beta_H^{-1}(g_1)) \otimes \beta_B(\beta_B(a_{2(0)(0)})(\alpha_H^{m+1}\beta_H^n(h_{121}) \triangleright \beta_B^{-1}(b_2)))_{(0)}) \otimes h_2g_2 \quad \text{(A4)} \\
 & a_1(\alpha_H^{m+p}\beta_H^{n+q}(a_{2(-1)})) \triangleright \beta_B^{-1}(\alpha_H^m\beta_H^n(h_{11}) \triangleright \beta_B^{-1}(b_1))) \otimes (\alpha_H^p\beta_H^{q+1}(a_{2(0)(-1)})(\alpha_H^{m+1}\beta_H^n(h_{121}) \triangleright \\
 & \beta_B^{-1}(b_2)))_{(-1)}(h_{122}))g_1 \otimes \beta_B^2(a_{2(0)(0)})\beta_B((\alpha_H^{m+1}\beta_H^n(h_{121}) \triangleright \beta_B^{-1}(b_2)))_{(0)}) \otimes h_2g_2 \quad \text{(R5)} \\
 & (\alpha_H^m\beta_H^{n-1}(h_{11}) \triangleright \beta_B^{-2}(b_1))) \otimes (\alpha_H^p\beta_H^{q+1}(a_{2(0)(-1)})\alpha_H^{-1}(\alpha_H(h_{121})\alpha_H^{q+1}\beta_H^{-1}(b_{2(-1)})))g_1 \otimes \\
 & \beta_B^2(a_{2(0)(0)})(\alpha_H^m\beta_H^{n+1}(h_{122}) \triangleright b_{2(0)}) \otimes h_2g_2 \quad \text{(CM4)} \\
 & (\alpha_H^m\beta_H^{n-1}(h_{11}) \triangleright \beta_B^{-2}(b_1))) \otimes (\alpha_H^p\beta_H^{q+1}(a_{2(-1)2})\alpha_H^{-1}(\alpha_H(h_{121})\alpha_H^{p+1}\beta_H^{q-1}(b_{2(-1)})))g_1 \otimes \\
 & \beta_B(a_{2(0)})(\alpha_H^m\beta_H^{n+1}(h_{122}) \triangleright b_{2(0)}) \otimes h_2g_2 \quad \text{(M4)} \\
 & (\alpha_H^m\beta_H^{n-1}(h_{11}) \triangleright \beta_B^{-2}(b_1))) \otimes (\alpha_H^p\beta_H^{q+1}(a_{2(-1)2})\alpha_H^{-1}(\alpha_H(h_{121})\alpha_H^{p+1}\beta_H^{q-1}(b_{2(-1)})))g_1 \otimes \\
 & \beta_B^{-1}(b_1)) \otimes (\alpha_H^p\beta_H^{q+1}(a_{2(-1)2})(h_{121}\alpha_H^p\beta_H^{q-1}(b_{2(-1)})))g_1 \otimes \beta_B(a_{2(0)})(\alpha_H^m\beta_H^{n+1}(h_{122}) \triangleright b_{2(0)}) \otimes h_2g_2 \quad \text{(C4)} \\
 & a_1((\alpha_H^{m+p}\beta_H^{n+q}(a_{2(-1)1})\alpha_H^m\beta_H^{n-1}(h_{11})) \triangleright \beta_B^{-1}(b_1)) \otimes (\alpha_H^p\beta_H^{q+1}(a_{2(-1)2})(\alpha_H^{-1}(h_{12})\alpha_H^p\beta_H^{q-1}(b_{2(-1)})))g_1 \otimes \\
 & \beta_B(a_{2(0)})(\alpha_H^m\beta_H^n(h_{21}) \triangleright b_{2(0)}) \otimes \beta_H(h_{22})g_2 \quad \text{(A4)} \\
 & a_1((\alpha_H^{m+p}\beta_H^{n+q}(a_{2(-1)1})\alpha_H^m\beta_H^{n-1}(h_{11})) \triangleright \beta_B^{-1}(b_1)) \otimes (\alpha_H^p\beta_H^{q+1}(a_{2(-1)2})h_{12})(\alpha_H^p\beta_H^q(b_{2(-1)})\beta_H^{-1}(g_1)) \otimes \\
 & \beta_B(a_{2(0)})(\alpha_H^m\beta_H^n(h_{21}) \triangleright b_{2(0)}) \otimes \beta_H(h_{22})g_2 = \Delta_{B \times_q^l H}(a \otimes h)\Delta_{B \times_q^l H}(b \otimes g).
 \end{aligned}$$

剩下的易证.证毕.

注记 7 (1)当 $m=1, n=-1, p=-1, q=1$ 时可得文献[6]中定理 1.

(2)当 $\alpha=\beta, m=0, n=0, p=0, q=0$ 时可得文献[4]中定理 3.5.

(3)当 $\alpha=\beta, n=0, q=0$ 时可得文献[5]中定理 3.6.

命题 3 设 $(B_{\times_{n,q}^{m,p}}H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 是广义双积 MBH 双代数, H 是带有对极 S_H 的 MBH-Hopf 代数, 在 $Hom(B, B)$ 中 $S_B : B \rightarrow B$ 是 Id_B 的卷积逆, 并且满足 $S_B \circ \alpha_B = \alpha_B \circ S_B, S_B \circ \beta_B = \beta_B \circ S_B$, 则 $(B_{\times_{n,q}^{m,p}}H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 是 MBH-Hopf 代数. 对极 S 定义为

$$S(b \otimes h) = (1_B \otimes S_H(\alpha_H^p\beta_H^{q-1}(b_{(-1)})\alpha_H^{-1}\beta_H^{-1}(h)))(S_B(b_{(0)}) \otimes 1_H),$$

其中 $b \in B, h \in H, m, n, p, q \in \mathbb{Z}$.

证明 按照定义直接可证.

注记 8 (1)当 $m=1, n=-1, p=-1, q=1$ 时可得文献[6]中命题 3, 其中对极 S 定义应为 $S(b \otimes h) = (1_B \otimes S_H(\alpha_H^{-1}(b_{(-1)})\alpha_H^{-1}\beta_H^{-1}(h)))(S_B(b_{(0)}) \otimes 1_H)$.

(2)当 $\alpha=\beta, m=0, n=0, p=0, q=0$ 时可得文献[4]中命题 3.6.

(3)当 $\alpha=\beta, n=0, q=0$ 时可得文献[5]中命题 3.8.

3 MBH-Yetter-Drinfeld 范畴

定义 10 设 $(H, \alpha_H, \beta_H, S)$ 是 MBH-Hopf 代数, $(U, \triangleright, \alpha_U, \beta_U)$ 是左 H -模并且 $(U, \rho, \alpha_U, \beta_U)$ 是左 H -余模(写作 $\rho(u) = u_{(-1)} \otimes u_{(0)}$). 如果下面条件成立:

$$(HYD) (\alpha_H^k\beta_H^l(g_1) \triangleright u)_{(-1)}g_2 \otimes (\alpha_H^k\beta_H^l(g_1) \triangleright u)_{(0)} = g_1\alpha_H(u_{(-1)}) \otimes \alpha_H^{k-1}\beta_H^l(g_2) \triangleright u_{(0)},$$

其中 $g \in H, u \in U, k, \ell \in \mathbb{Z}$, 则 $(U, \triangleright, \rho, \alpha_U, \beta_U)$ 是(左-左)MBH-Yetter-Drinfeld 模. MBH-Yetter-Drinfeld 模范畴用 ${}^H_H \text{HYD}^{(k, \ell)}$ 表示.

注记 9 (1)(HYD)条件等价于(R5), i.e.,

$$\alpha_H^k \beta_H^{q-1} ((\alpha_H^m \beta_H^{n+1}(h_1) \triangleright u)_{(-1)}) h_2 \otimes (\alpha_H^m \beta_H^{n+1}(h_1) \triangleright u)_{(0)} = h_1 \alpha_H^{k+1} \beta_H^{q-1}(u_{(-1)}) \otimes \alpha_H^{m-1} \beta_H^{n+1}(h_2) \triangleright u_{(0)}.$$

(2)当 $\alpha = \beta, m = 0, n = 0, p = 0, q = 0$ 时可得文献[4]中定义 4.1.

(3)当 $\alpha = \beta, n = 0, q = 0$ 时可得文献[5]中定义 4.1.

下面结果易证:

命题 4 等式(HYD)等价于下面形式:

$$(\text{HYD})' \quad \rho(\alpha_H^k \beta_H^\ell(g) \triangleright u) = (g_{11} \alpha_H^{-1}(u_{(-1)})) S(g_2) \otimes \alpha_H^{k+1} \beta_H^\ell(g_{12}) \triangleright u_{(0)},$$

其中 $g \in H, u \in U$ 和 $k, \ell \in \mathbb{Z}$.

定理 2 如果 $(U, \alpha_U, \beta_U), (V, \alpha_V, \beta_V)$ 是 MBH-Yetter-Drinfeld 模, 则 $(U \otimes V, \alpha_{U \otimes V} = \alpha_U \otimes \alpha_V, \beta_{U \otimes V} = \beta_U \otimes \beta_V)$ 是 MBH-Yetter-Drinfeld 模, 其中模作用和余模作用分别是:

$$\begin{aligned} h \triangleright (u \otimes v) &= h_1 \triangleright u \otimes h_2 \triangleright v, \\ \rho(u \otimes v) &= u_{(-1)} v_{(-1)} \otimes u_{(0)} \otimes v_{(0)}, \end{aligned}$$

其中 $u \in U, v \in V$ 和 $h \in H$.

证明 下面只验证条件(HYD)'成立, 其他证明省略.

$$\begin{aligned} \rho(\alpha_H^k \beta_H^\ell(h) \triangleright (u \otimes v)) &= \rho(\alpha_H^k \beta_H^\ell(h_1) \triangleright u \otimes \alpha_H^k \beta_H^\ell(h_2) \triangleright v) = (\alpha_H^k \beta_H^\ell(h_1) \triangleright u)_{(-1)} (\alpha_H^k \beta_H^\ell(h_2) \triangleright v)_{(-1)} \otimes \\ &(\alpha_H^k \beta_H^\ell(h_1) \triangleright u)_{(0)} \otimes (\alpha_H^k \beta_H^\ell(h_2) \triangleright v)_{(0)} \stackrel{(\text{HYD})'}{=} ((h_{111} \alpha_H^{-1}(u_{(-1)})) S(h_{12})) ((h_{211} \alpha_H^{-1}(v_{(-1)})) S(h_{22})) \otimes \\ &(\alpha_H^{k+1} \beta_H^\ell(h_{112}) \triangleright u_{(0)}) \otimes (\alpha_H^{k+1} \beta_H^\ell(h_{212}) \triangleright v_{(0)}) \stackrel{(C4)}{=} ((h_{111} \alpha_H^{-1}(u_{(-1)})) S(\alpha_H(h_{121}))) ((\alpha_H^{-1} \beta_H(h_{122}) \\ &\alpha_H^{-1}(v_{(-1)})) S(h_{22})) \otimes (\alpha_H^{k+1} \beta_H^\ell(h_{112}) \triangleright u_{(0)}) \otimes (\alpha_H^{k+1} \beta_H^{\ell-1}(h_{21}) \triangleright v_{(0)}) \stackrel{(A4)}{=} (\alpha_H(h_{111}) u_{(-1)}) (S(\alpha_H(h_{121}))) \\ &((\alpha_H^{-1}(h_{122}) \alpha_H^{-1} \beta_H^{-1}(v_{(-1)})) S(\beta_H^{-1}(h_{22}))) \otimes (\alpha_H^{k+1} \beta_H^\ell(h_{112}) \triangleright u_{(0)}) \otimes (\alpha_H^{k+1} \beta_H^{\ell-1}(h_{21}) \triangleright v_{(0)}) \stackrel{(A4)}{=} \\ &(\alpha_H(h_{111}) u_{(-1)}) ((S(\alpha_H^{-1}(h_{121})) \alpha_H^{-1}(h_{122})) \alpha_H^{-1}(v_{(-1)})) S(h_{22})) \otimes (\alpha_H^{k+1} \beta_H^\ell(h_{112}) \triangleright u_{(0)}) \otimes \\ &(\alpha_H^{k+1} \beta_H^{\ell-1}(h_{21}) \triangleright v_{(0)}) \stackrel{(D5)}{=} (\alpha_H(h_{111}) u_{(-1)}) ((\epsilon(h_{12}) \alpha_H^{-1} \beta_H(v_{(-1)})) S(h_{22})) \otimes (\alpha_H^{k+1} \beta_H^\ell(h_{112}) \triangleright \\ &u_{(0)}) \otimes (\alpha_H^{k+1} \beta_H^{\ell-1}(h_{21}) \triangleright v_{(0)}) \stackrel{(C4)}{=} (\alpha_H(h_{111}) u_{(-1)}) ((\epsilon(h_{121}) \alpha_H^{-1} \beta_H(v_{(-1)})) S(\beta_H^{-1}(h_2))) \otimes \\ &(\alpha_H^{k+1} \beta_H^\ell(h_{112}) \triangleright u_{(0)}) \otimes (\alpha_H^{k+1} \beta_H^\ell(h_{122}) \triangleright v_{(0)}) \stackrel{(C5)}{=} (\alpha_H(h_{111}) u_{(-1)}) (\alpha_H^{-1} \beta_H(v_{(-1)}) S(\beta_H^{-1}(h_2))) \otimes \\ &(\alpha_H^{k+1} \beta_H^\ell(h_{112}) \triangleright u_{(0)}) \otimes (\alpha_H^{k+1} \beta_H^{\ell-1}(h_{12}) \triangleright v_{(0)}) \stackrel{(A4)}{=} (\alpha_H(h_{111}) \alpha_H^{-1}(u_{(-1)} v_{(-1)})) S(h_2) \otimes (\alpha_H^{k+1} \beta_H^\ell(h_{112}) \triangleright \\ &u_{(0)}) \otimes (\alpha_H^{k+1} \beta_H^{\ell-1}(h_{12}) \triangleright v_{(0)}) \stackrel{(C4)}{=} (h_{11}) \alpha_H^{-1}(u_{(-1)} v_{(-1)}) S(h_2) \otimes (\alpha_H^{k+1} \beta_H^\ell(h_{121}) \triangleright u_{(0)}) \otimes \\ &(\alpha_H^{k+1} \beta_H^\ell(h_{122}) \triangleright v_{(0)}) = (h_{11}) \alpha_H^{-1}(u_{(-1)} v_{(-1)}) S(h_2) \otimes (\alpha_H^{k+1} \beta_H^\ell(h_{12}) \triangleright (u_{(0)} \otimes v_{(0)})). \end{aligned}$$

命题 5 设 $(H, \mu_H, \Delta_H, \alpha_H, \beta_H)$ 是 MBH 双代数, 并且 $(U, \alpha_U, \beta_U), (V, \alpha_V, \beta_V), (W, \alpha_W, \beta_W)$ 是 MBH-Yetter-Drinfeld 模. 定义线性映射:

$$\alpha_{U, V, W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W), (u \otimes v) \otimes w \mapsto \alpha_U(u) \otimes (v \otimes \beta_W^{-1}(w)),$$

则 $\alpha_{U, V, W}$ 是左 H -模和左 H -余模同态.

证明 直接验证可得.

注记 10 设 (H, α, β) 是 MBH-Hopf 代数. 则 (K, Id, Id) 是 MBH-Yetter-Drinfeld 模, 其中左 H -模 $H \otimes K \rightarrow K, h \otimes t \mapsto \epsilon(h)t$, 和左 H -余模 $K \rightarrow H \otimes K, t \mapsto 1_H \otimes t$.

命题 6 MBH-Yetter-Drinfeld 模范畴 ${}^H_H \text{HYD}^{(k, \ell)}$ 是张量范畴, 其中张量结构和单位 $I = (K, Id, Id)$ 分别在定理 2 和注 10 中给出. 对任意的 $(U, \alpha_U, \beta_U), (V, \alpha_V, \beta_V), (W, \alpha_W, \beta_W) \in {}^H_H \text{HYD}^{(k, \ell)}$, 其中结合子和单位限制分别为

$$\alpha_{U, V, W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W), (u \otimes v) \otimes w \mapsto \alpha_U(u) \otimes (v \otimes \beta_W^{-1}(w)),$$

$$l_V : K \otimes V \rightarrow V, t \otimes v \mapsto t \beta_V(v), r_V : V \otimes K \rightarrow V, v \otimes t \mapsto t \alpha_V(v).$$

证明 直接验证可得.

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A class of generalized biproduct and related Yetter-Drinfeld category

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Abstract: In this article, firstly, we introduce the notions of a generalized smash product MBH-algebra $(B \#_n^m H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ and a generalized smash coproduct MBH-coalgebra $(B \times_q^p H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$. Furthermore, we get the necessary and sufficient conditions for $(B \#_n^m H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ and $(B \times_q^p H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ to be a MBH-bialgebras $(B \#_{\times n, q}^{m, p} H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$. At last, we provide a class of monoidal category which is related to $(B \#_{\times n, q}^{m, p} H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$.

Keywords: Radford biproduct; BiHom-algebra; Yetter-Drinfeld category