

Radford 双积 monoidal BiHom-Hopf 代数

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摘要:首先给出 smash 积 monoidal BiHom-代数 $(B \# H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 和 smash 余积 monoidal BiHom-余代数 $(B \times H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$, 进而得到 $(B \# H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 和 $(B \times H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 构成 monoidal BiHom-双代数的充分必要条件.

关键词:Radford 双积; Monoidal BiHom-Hopf 代数; Hopf 代数

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构造 Hopf 代数是 Hopf 代数理论中的一个重要研究内容. 1985 年文献[1]通过 smash 积和 smash 余积构造了一类 Hopf 代数, 称之为 Radford 双积, 等价于构造了一个 Yetter-Drinfeld 范畴中的 Hopf 代数, 使得该结构在有限维有限 Hopf 代数的分类中起到了关键作用. 对 Radford 双积的其他研究可见文献[2-6].

Hom 型代数起源于文献[7]对 Witt 代数和 Virasoro 代数的 q -形变的研究时提出的 Hom-Lie 代数, 这是近年来 Hopf 代数理论的一个研究热点, 受到许多研究者的关注, 见文献[6-14]. 其中文献[13]从范畴角度研究 Hom 代数, 从而引入了 monoidal Hom-Hopf 代数的概念. 2015 年, 文献[14]又得到了这一结构的 monoidal BiHom 型, 并且给出了一类 BiHom smash 积结构. 2014 年, 在文献[6]给出了一种 Radford 双积的 monoidal Hom-型. 很自然地要问 Radford 双积的 monoidal Bi-Hom 型是什么? 本文主要给出 Radford 双积的一类 monoidal BiHom 型, 推广了文献[6]中的主要结果.

文中所有代数系统都是定义在域 K 上的. 若 C 是余代数, 则用 Sweedler 记号(文献[15])表示余乘: $\Delta(c) = c_1 \otimes c_2, \forall c \in C$. 记左 H -余模范畴为 ${}^H\text{Mod}$, 即 $(M, \rho) \in {}^H\text{Mod}, \rho(x) = x_{(-1)} \otimes x_{(0)} \in H \otimes M, \forall x \in M$. 给定一个 K -空间 M , 记 M 上恒等态射为 Id_M .

1 基本定义

以下给出文献[14]中的定义 monoidal 版本.

定义 1 Monoidal BiHom-代数是一个五元组 $(A, \mu, 1_A, \alpha, \beta)$ (缩写为 (A, α, β)), 其中 A 是一个 K -线性空间, $\mu: A \otimes A \rightarrow A$ 是一个 K -线性映射记 $\mu(a \otimes a') = aa', 1_A \in A, \alpha, \beta$ 是 A 的自同构, 对任意的 $a, a', a'' \in A$ 且满足下面条件:

- (A1) $\alpha \circ \beta = \beta \circ \alpha,$
- (A2) $\alpha(aa') = \alpha(a)\alpha(a'); \alpha(1_A) = 1_A,$
- (A3) $\beta(aa') = \beta(a)\beta(a'); \beta(1_A) = 1_A,$
- (A4) $\alpha(a)(a'a'') = (aa')\beta(a''),$

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(A5) $a1_A = \alpha(a); 1_{Aa} = \beta(a)$.

注1 当 $\alpha = \beta$ 时, Monoidal BiHom-代数是 Monoidal Hom-代数.

定义2 Monoidal BiHom-余代数是一个五元组 $(C, \Delta, \varepsilon, \psi, \omega)$ (缩写为 (C, ψ, ω)), 其中 C 是一个 K -线性空间, $\Delta: C \rightarrow C \otimes C, \varepsilon: C \rightarrow K$ 是一个 K -线性映射记 $\Delta(c) = c_1 \otimes c_2, \psi, \omega$ 是 C 的自同构, 对任意的 $c \in C$ 且满足下面条件:

(C1) $\psi \circ \omega = \omega \circ \psi,$

(C2) $\psi(c)_1 \otimes \psi(c)_2 = \psi(c_1) \otimes \psi(c_2); \varepsilon \circ \psi = \varepsilon,$

(C3) $\omega(c)_1 \otimes \omega(c)_2 = \omega(c_1) \otimes \omega(c_2); \varepsilon \circ \omega = \varepsilon,$

(C4) $\psi^{-1}(c_1) \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes \omega^{-1}(c_2),$

(C5) $\varepsilon(c_1)c_2 = \omega^{-1}(c); c_1\varepsilon(c_2) = \psi^{-1}(c).$

注2 当 $\psi = \omega$ 时, Monoidal BiHom-余代数是 Monoidal Hom-余代数.

定义3 Monoidal BiHom-双代数是一个7元组 $(H, \mu, 1_H, \Delta, \varepsilon, \alpha, \beta)$ (缩写为 (H, α, β)), 其中 $(H, \mu, 1_H, \alpha, \beta)$ 是一个 monoidal BiHom-代数, $(H, \Delta, \varepsilon, \alpha, \beta)$ 是一个 monoidal BiHom-余代数, 并且使得满足

$$\Delta(hh') = \Delta(h)\Delta(h'); \Delta(1_H) = 1_H \otimes 1_H, \varepsilon(hh') = \varepsilon(h)\varepsilon(h'); \varepsilon(1_H) = 1.$$

此外, 如果存在线性映射 $S: H \rightarrow H$ 使得

$$S(h_1)h_2 = h_1S(h_2) = \varepsilon(h)1_H, S \circ \alpha = \alpha \circ S, \text{ 和 } S \circ \beta = \beta \circ S,$$

则称 $(H, \mu, 1_H, \Delta, \varepsilon, \alpha, \beta, S)$ (abbr. (H, α, β, S)) 是 Monoidal BiHom-Hopf 代数.

定义4 设 (A, α_A, β_A) 是 monoidal BiHom-代数. 一个左 A -模是一个四元组 $(M, \triangleright, \alpha_M, \beta_M)$, 其中 M 是一个线性空间, $\triangleright: A \otimes M \rightarrow M$ 是线性映射, 且 α_M, β_M 是 M 的自同构, 对任意的 $a, a' \in A$ 和 $m \in M$ 使得

(M1) $\alpha_M \circ \beta_M = \beta_M \circ \alpha_M,$

(M2) $\alpha_M(a \triangleright m) = \alpha_A(a) \triangleright \alpha_M(m),$

(M3) $\beta_M(a \triangleright m) = \beta_A(a) \triangleright \beta_M(m),$

(M4) $\alpha_A(a) \triangleright (a' \triangleright m) = (aa') \triangleright \beta_M(m); 1_A \triangleright m = \beta_M(m).$

注3 显然 $(A, \mu, \alpha_A, \beta_A)$ 是左 (A, α_A, β_A) -模.

定义5 设 (C, ψ_C, ω_C) 是一个 monoidal BiHom-余代数. 一个左 C -余模是三元组 (M, ψ_M, ω_M) , 其中 M 是一个线性空间, $\rho: M \rightarrow C \otimes M$ (记 $\rho(m) = m_{(-1)} \otimes m_{(0)}, m \in M$) 是线性映射, 且 ψ_M, ω_M 是 M 的自同构, 对任意的 $m \in M$ 使得

(CM1) $\psi_M \circ \omega_M = \omega_M \circ \psi_M,$

(CM2) $\psi_M(m)_{(-1)} \otimes \psi_M(m)_{(0)} = \psi_C(m_{(-1)}) \otimes \psi_M(m_{(0)}),$

(CM3) $\omega_M(m)_{(-1)} \otimes \omega_M(m)_{(0)} = \omega_C(m_{(-1)}) \otimes \omega_M(m_{(0)}),$

(CM4) $\psi_C^{-1}(m_{(-1)}) \otimes m_{(0)(-1)} \otimes m_{(0)(0)} = m_{(-1)1} \otimes m_{(-1)2} \otimes \omega_M^{-1}(m_{(0)});$

$$\varepsilon_C(m_{(-1)})m_{(0)} = \omega_M^{-1}(m).$$

注4 显然 $(C, \Delta, \psi_C, \omega_C)$ 是左 $-(C, \psi_C, \omega_C)$ 余模.

定义6 设 (H, α_H, β_H) 是 monoidal BiHom-双代数. 一个 monoidal BiHom-代数 (B, α_B, β_B) 称为左 $-H$ 模 monoidal BiHom-代数, 如果 (B, α_B, β_B) 是一个有 \triangleright 作用的左 $-H$ 模, 并且对任意的 $b, b' \in B; h \in H$ 满足下面条件:

$$h \triangleright (bb') = (h_1 \triangleright b)(h_2 \triangleright b'), h \triangleright 1_B = \varepsilon(h)1_B.$$

定义7 设 (H, α_H, β_H) 是 monoidal BiHom-双代数. 一个 monoidal BiHom-余代数 (B, α_B, β_B) 称为左 $-H$ 余模 monoidal BiHom-余代数, 如果 (B, α_B, β_B) 是一个左 $-H$ 余模(余作用为 $\rho(b) = b_{(-1)} \otimes b_{(0)}$), 并且对任意的 $b \in B$ 满足下面条件:

$$b_{(-1)} \otimes \Delta_B(b_{(0)}) = b_{1(-1)}b_{2(-1)} \otimes b_{1(0)} \otimes b_{2(0)}, b_{(-1)}\varepsilon(b_{(0)}) = \varepsilon(b)1_H.$$

定义8 设 (H, α_H, β_H) 是 monoidal BiHom-双代数. 一个 monoidal BiHom-代数 (B, α_B, β_B) 称为左 $-H$ 余模 monoidal BiHom-代数, 如果 (B, α_B, β_B) 是一个左 $-H$ 余模且余作用为 ρ , 并且对任意的 $a, b \in B$ 满足下面条件:

$$\rho(ab) = a_{(-1)}b_{(-1)} \otimes a_{(0)}b_{(0)}, \rho(1_B) = 1_H \otimes 1_B.$$

定义 9 设 (H, α_H, β_H) 是 monoidal BiHom- 双代数. 一个 monoidal BiHom- 余代数 (B, α_B, β_B) 称为左 $-H$ 模 monoidal BiHom- 余代数, 如果 (B, α_B, β_B) 是一个有 \triangleright 作用的左 $-H$ 模, 并且对任意的 $b \in B; h \in H$ 满足下面条件:

$$\Delta(h \triangleright b) = h_1 \triangleright b_1 \otimes h_2 \triangleright b_2, \varepsilon_B(h \triangleright b) = \varepsilon_H(h)\varepsilon_B(b).$$

2 主要结果

命题 1 设 (H, α_H, β_H) 是 monoidal BiHom- 双代数, (B, α_B, β_B) 是左 H -BiHom- 模代数, 则 $(B \# H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ ($= B \otimes H$ 作为向量空间) 是一个 monoidal BiHom- 代数, 单位元为 $1_B \otimes 1_H$ 且乘法为

$$(a \# h)(b \# g) = a(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)) \# \beta_H(h_2)g,$$

其中 $a, b \in B, h, g \in H$. 这种情况下, 称 $(B \# H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 为 smash 积 monoidal BiHom- 代数.

证明 显然可得到 $(\alpha_B \otimes \alpha_H) \circ (\beta_B \otimes \beta_H) = (\beta_B \otimes \beta_H) \circ (\alpha_B \otimes \alpha_H)$,

$$(\alpha_B \otimes \alpha_H)(1_B \otimes 1_H) = 1_B \otimes 1_H \text{ 和 } (\beta_B \otimes \beta_H)(1_B \otimes 1_H) = 1_B \otimes 1_H.$$

$$(a \otimes h)(1_B \otimes 1_H) = a(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(1_B)) \otimes \beta_H(h_2)1_H = \alpha(\beta_H^{-1}(\alpha_H(h_1)) \triangleright 1_B) \otimes \alpha_H(\beta_H(h_2)) = a1_B\varepsilon(h_1) \otimes \alpha_H(\beta_H(h_2)) = (\alpha_B \otimes \alpha_H)(a \otimes h).$$

类似可得

$$(1_B \otimes 1_H)(a \otimes h) = (\beta_B \otimes \beta_H)(a \otimes h).$$

下证: $(\alpha_B \otimes \alpha_H)((a \otimes h)(b \otimes g)) = ((\alpha_B \otimes \alpha_H)(a \otimes h))((\alpha_B \otimes \alpha_H)(b \otimes g))$. 事实上,

$$\begin{aligned} (\alpha_B \otimes \alpha_H)((a \otimes h)(b \otimes g)) &= (\alpha_B \otimes \alpha_H)(a(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)) \otimes \beta_H(h_2)g) = \\ &= \alpha_B(a(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b))) \otimes \alpha_H(\beta_H(h_2)g) = \alpha_B(a)\alpha_B(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)) \otimes \alpha_H(\beta_H(h_2))\alpha_H(g) = \\ &= \alpha_B(a)(\beta_H^{-1}(\alpha_H^2(h_1)) \triangleright \alpha_B(\beta_B^{-1}(b))) \otimes \alpha_H(\beta_H(h_2))\alpha_H(g) = ((\alpha_B \otimes \alpha_H)(a \otimes h))((\alpha_B \otimes \alpha_H)(b \otimes g)). \end{aligned}$$

类似可证:

$$(\beta_B \otimes \beta_H)((a \otimes h)(b \otimes g)) = ((\beta_B \otimes \beta_H)(a \otimes h))((\beta_B \otimes \beta_H)(b \otimes g)).$$

下证: $((\alpha_B \otimes \alpha_H)(a \otimes h))((b \otimes k)(c \otimes g)) = ((a \otimes h)(b \otimes k))((\beta_B \otimes \beta_H)(c \otimes g))$. 事实上,

$$\begin{aligned} ((\alpha_B \otimes \alpha_H)(a \otimes h))((b \otimes k)(c \otimes g)) &= (\alpha_B(a) \otimes \alpha_H(h))(b(\beta_H^{-1}(\alpha_H(k_1)) \triangleright \beta_B^{-1}(c)) \otimes \beta_H(k_2)g) = \\ &= \alpha_B(a)(\beta_H^{-1}(\alpha_H(\alpha_H(h_1))) \triangleright \beta_B^{-1}(b(\beta_H^{-1}(\alpha_H(k_1)) \triangleright \beta_B^{-1}(c)))) \otimes \beta_H(\alpha_H(h_2))(\beta_H(k_2)g) = \\ &= \alpha_B(a)(\beta_H^{-1}(\alpha_H^2(h_1)) \triangleright (\beta_B^{-1}(b)(\beta_H^{-1}(\alpha_H(k_1)) \triangleright \beta_B^{-1}(c)))) \otimes \beta_H(\alpha_H(h_2))(\beta_H(k_2)g) = \\ &= \alpha_B(a)((\beta_H^{-1}(\alpha_H^2(h_1)))_1 \triangleright \beta_B^{-1}(b))(\beta_H^{-1}(\alpha_H^2(h_1))_2 \triangleright (\beta_H^{-1}(\alpha_H(k_1)) \triangleright \beta_B^{-1}(c))) \otimes \beta_H(\alpha_H(h_2))(\beta_H(k_2)g) = \\ &= \alpha_B(a)((\beta_H^{-1}(\alpha_H^2(h_{11}))) \triangleright \beta_B^{-1}(b))(\beta_H^{-1}(\alpha_H^2(h_{12})) \triangleright (\beta_H^{-1}(\alpha_H(k_1)) \triangleright \beta_B^{-1}(c))) \otimes \beta_H(\alpha_H(h_2))(\beta_H(k_2)g). \end{aligned}$$

和

$$\begin{aligned} ((a \otimes h)(b \otimes k))((\beta_B \otimes \beta_H)(c \otimes g)) &= (a(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)) \otimes \beta_H(h_2)k)(\beta_B(c) \otimes \beta_H(g)) = \\ &= (a(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)))(\beta_H^{-1}(\alpha_H((\beta_H(h_2)k)_1)) \triangleright c) \otimes \beta_H((\beta_H(h_2)k)_2)\beta_H(g) = (\alpha_H(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)))(\beta_H^{-1}(\alpha_H(\beta_H(h_{21})k_1)) \triangleright c) \otimes \beta_H(\beta_H(h_{22})k_2)\beta_H(g) = \alpha_B(a)((\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)))(\beta_H^{-1}(\alpha_H(h_{21}))\beta_H^2(\alpha_H(k_1)) \triangleright \beta_B^{-1}(c)) \otimes (\beta_H^2(h_{22})\beta_H(k_2))\beta_H(g) = \alpha_B(a)((\beta_H^{-1}(\alpha_H^2(h_{11}))) \triangleright \beta_B^{-1}(b))(\beta_H^{-1}(\alpha_H(h_{12}))\beta_H^2(\alpha_H(k_1)) \triangleright \beta_B^{-1}(c)) \otimes (\beta_H(h_2)\beta_H(k_2))\beta_H(g) = \alpha_B(a)((\beta_H^{-1}(\alpha_H^2(h_{11}))) \triangleright \beta_B^{-1}(b))(\beta_H^{-1}(\alpha_H(h_{12}))\beta_H^2(\alpha_H(k_1)) \triangleright \beta_B^{-1}(c)) \otimes \beta_H(\alpha_H(h_2))(\beta_H(k_2)g). \end{aligned}$$

证毕.

命题 2 设 (H, α_H, β_H) 是 monoidal BiHom- 双代数, (B, α_B, β_B) 是左 H -BiHom- 余模余代数, 则 $(B \times H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ ($= B \otimes H$ 作为向量空间) 是一个 monoidal BiHom- 余代数, 余单位元为 $\varepsilon_B \otimes \varepsilon_H$ 且余乘为 $\Delta(b \otimes h) = b_1 \otimes \alpha_H^{-1}(\beta_H(b_{2(-1)}))\beta_H^{-1}(h_1) \otimes \beta_B(b_{2(0)}) \otimes h_2$, 其中 $b \in B, h \in H$. 这种情况下, 称 $(B \times H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 为 smash 余积 monoidal BiHom- 余代数.

证明 显然可得到 $(\alpha_B \otimes \alpha_H) \circ (\beta_B \otimes \beta_H) = (\beta_B \otimes \beta_H) \circ (\alpha_B \otimes \alpha_H)$, $(\epsilon_B \otimes \epsilon_H) \circ (\alpha_B \otimes \alpha_H) = \epsilon_B \otimes \epsilon_H$ 和 $(\epsilon_B \otimes \epsilon_H) \circ (\beta_B \otimes \beta_H) = \epsilon_B \otimes \epsilon_H$.

下证: $(Id_{B \otimes H} \otimes \epsilon_{B \otimes H}) \Delta(b \otimes h) = (\alpha_B^{-1} \otimes \alpha_H^{-1})(b \otimes h)$.

$$(Id_{B \otimes H} \otimes \epsilon_{B \otimes H}) \Delta(b \otimes h) = (Id_{B \otimes H} \otimes \epsilon_{B \otimes H})(b_1 \otimes \alpha_H^{-1}(\beta_H(b_{2(-1)})))\beta_H^{-1}(h_1) \otimes \beta_B(b_{2(0)}) \otimes h_2 = b_1 \otimes \alpha_H^{-1}(\beta_H(b_{2(-1)}))\beta_H^{-1}(h_1)\epsilon(b_{2(0)})\epsilon(h_2) = (\alpha_B^{-1} \otimes \alpha_H^{-1})(b \otimes h).$$

类似可得: $(\epsilon_{B \otimes H} \otimes Id_{B \otimes H}) \Delta(b \otimes h) = (\beta_B^{-1} \otimes \beta_H^{-1})(b \otimes h)$.

下证:

$$((\beta_B \otimes \beta_H) \otimes (\beta_B \otimes \beta_H)) \Delta(b \otimes h) = \Delta((\beta_B \otimes \beta_H)(b \otimes h)), ((\beta_B \otimes \beta_H) \otimes (\beta_B \otimes \beta_H)) \Delta(b \otimes h) = ((\beta_B \otimes \beta_H) \otimes (\beta_B \otimes \beta_H))(b_1 \otimes \alpha_H^{-1}(\beta_H(b_{2(-1)})))\beta_H^{-1}(h_1) \otimes \beta_B(b_{2(0)}) \otimes h_2 = \beta_B(b_1) \otimes \alpha_H^{-1}(\beta_H^2(b_{2(-1)}))h_1 \otimes \beta_B^2(b_{2(0)}) \otimes \beta_H(h_2) = \Delta((\beta_B \otimes \beta_H)(b \otimes h)).$$

类似可证: $((\alpha_B \otimes \alpha_H) \otimes (\alpha_B \otimes \alpha_H)) \Delta(b \otimes h) = \Delta((\alpha_B \otimes \alpha_H)(b \otimes h))$.

下证:

$$(\Delta \otimes (\beta_B \otimes \beta_H)^{-1}) \Delta(b \otimes h) = ((\alpha_B \otimes \alpha_H)^{-1} \otimes \Delta) \Delta(b \otimes h).$$

$$\begin{aligned} (\Delta \otimes (\beta_B \otimes \beta_H)^{-1}) \Delta(b \otimes h) &= (\Delta \otimes \beta_B^{-1} \otimes \beta_H^{-1})(b_1 \otimes \alpha_H^{-1}(\beta_H(b_{2(-1)})))\beta_H^{-1}(h_1) \otimes \beta_B(b_{2(0)}) \otimes h_2 = \\ &= \Delta(b_1 \otimes \alpha_H^{-1}(\beta_H(b_{2(-1)})))\beta_H^{-1}(h_1) \otimes b_{2(0)} \otimes \beta_H^{-1}(h_2) = b_{11} \otimes \alpha_H^{-1}(\beta_H(b_{12(-1)}))\beta_H^{-1}((\alpha_H^{-1}(\beta_H(b_{2(-1)}))) \cdot \\ & \quad \beta_H^{-1}(h_1))_1) \otimes \beta_B(b_{12(0)}) \otimes (\alpha_H^{-1}(\beta_H(b_{2(-1)})))\beta_H^{-1}(h_1)_2 \otimes b_{2(0)} \otimes \beta_H^{-1}(h_2) = b_{11} \otimes \\ & \quad \alpha_H^{-1}(\beta_H(b_{12(-1)}))\beta_H^{-1}(\alpha_H^{-1}(\beta_H(b_{2(-1)})))\beta_H^{-1}(h_{11})) \otimes \beta_B(b_{12(0)}) \otimes \alpha_H^{-1}(\beta_H(b_{2(-1)2}))\beta_H^{-1}(h_{12}) \otimes \\ & \quad b_{2(0)} \otimes \beta_H^{-1}(h_2) = \alpha_B^{-1}(b_1) \otimes \alpha_H^{-1}(\beta_H(b_{21(-1)}))\beta_H^{-1}(\alpha_H^{-1}(\beta_H(\beta_B(b_{22(-1)1})))\beta_H^{-1}(h_{11})) \otimes \\ & \quad \beta_B(b_{21(0)}) \otimes \alpha_H^{-1}(\beta_H(\beta_B(b_{22(-1)2}))\beta_H^{-1}(h_{12})) \otimes \beta_B(b_{22(0)}) \otimes \beta_H^{-1}(h_2) = \\ & \quad \alpha_B^{-1}(b_1) \otimes \alpha_H^{-1}(\beta_H(b_{21(-1)}))(\alpha_H^{-1}(\beta_H(b_{22(-1)1}))\beta_H^{-1}(h_{11})) \otimes \beta_B(b_{21(0)}) \otimes \\ & \quad \alpha_H^{-1}(\beta_H^2(b_{22(-1)2}))\beta_H^{-1}(h_{12}) \otimes \beta_B(b_{22(0)}) \otimes \beta_H^{-1}(h_2) = \alpha_B^{-1}(b_1) \otimes \\ & \quad (\alpha_H^{-2}(\beta_H(b_{21(-1)}))\alpha_H^{-1}(\beta_H(b_{22(-1)1})))\beta_H^{-1}(h_{11}) \otimes \beta_B(b_{21(0)}) \otimes \\ & \quad \alpha_H^{-1}(\beta_H^2(b_{22(-1)2}))\beta_H^{-1}(h_{12}) \otimes \beta_B(b_{22(0)}) \otimes \beta_H^{-1}(h_2). \end{aligned}$$

又有

$$\begin{aligned} ((\alpha_B \otimes \alpha_H)^{-1} \otimes \Delta) \Delta(b \otimes h) &= (\alpha_B^{-1} \otimes \alpha_H^{-1} \otimes \Delta)(b_1 \otimes \alpha_H^{-1}(\beta_H(b_{2(-1)})))\beta_H^{-1}(h_1) \otimes \\ & \quad \beta_B(b_{2(0)}) \otimes h_2 = \alpha_B^{-1}(b_1) \otimes \alpha_H^{-2}(\beta_H(b_{2(-1)}))\alpha_H^{-1}(\beta_H^{-1}(h_1)) \otimes \Delta(\beta_B(b_{2(0)}) \otimes h_2) = \\ & \quad \alpha_B^{-1}(b_1) \otimes \alpha_H^{-2}(\beta_H(b_{2(-1)}))\alpha_H^{-1}(\beta_H^{-1}(h_1)) \otimes \beta_B(b_{2(0)1}) \otimes \alpha_H^{-1}(\beta_H(\beta_B(b_{2(0)2(-1)})))\beta_H^{-1}(h_{21}) \otimes \\ & \quad \beta_B(\beta_B(b_{2(0)2(0)})) \otimes h_{22} = \alpha_B^{-1}(b_1) \otimes \alpha_H^{-2}(\beta_H(b_{2(-1)}))\alpha_H^{-1}(\beta_H^{-1}(h_1)) \otimes \beta_B(b_{2(0)1}) \otimes \\ & \quad \alpha_H^{-1}(\beta_H^2(b_{2(0)2(-1)}))\beta_H^{-1}(h_{21}) \otimes \beta_B^2(b_{2(0)2(0)}) \otimes h_{22} = \alpha_B^{-1}(b_1) \otimes \alpha_H^{-2}(\beta_H(b_{21(-1)}b_{22(-1)})) \cdot \\ & \quad \alpha_H^{-1}(\beta_H^{-1}(h_1)) \otimes \beta_B(b_{21(0)}) \otimes \alpha_H^{-1}(\beta_H^2(b_{22(0)(-1)}))\beta_H^{-1}(h_{21}) \otimes \beta_B^2(b_{22(0)(0)}) \otimes h_{22} = \\ & \quad \alpha_B^{-1}(b_1) \otimes \alpha_H^{-2}(\beta_H(b_{21(-1)}b_{22(-1)}))\beta_H^{-1}(h_{11}) \otimes \beta_B(b_{21(0)}) \otimes \alpha_H^{-1}(\beta_H^2(b_{22(0)(-1)}))\beta_H^{-1}(h_{12}) \otimes \\ & \quad \beta_B^2(b_{22(0)(0)}) \otimes \beta_H^{-1}(h_2) = \alpha_B^{-1}(b_1) \otimes (\alpha_H^{-2}(\beta_H(b_{21(-1)}))\alpha_H^{-1}(\beta_H(b_{22(-1)1})))\beta_H^{-1}(h_{11}) \otimes \\ & \quad \beta_B(b_{21(0)}) \otimes \alpha_H^{-1}(\beta_H^2(b_{22(-1)2}))\beta_H^{-1}(h_{12}) \otimes \beta_B(b_{22(0)}) \otimes \beta_H^{-1}(h_2). \end{aligned}$$

证毕.

注 5 当 $\alpha_H = \beta_H, \alpha_B = \beta_B$ 时, 此命题是文献[6]中的命题 3. 2.

定理 1 设 (H, α_H, β_H) 是 monoidal BiHom- 双代数, $(B \# H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 是 smash 积 monoidal BiHom- 代数, $(B \times H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 是 smash 余积 monoidal BiHom- 余代数, 则下面条件等价:

(i) $(B \# H, m_{B \# H}, 1_B \otimes 1_H, \Delta_{B \times H}, \epsilon_B \otimes \epsilon_H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 是 monoidal BiHom- 双代数.

(ii) 下面条件成立:

(c₁) ϵ_B 是代数同态, $\Delta_B(1_B) = 1_B \otimes 1_B$,

(c₂) (B, α_B, β_B) 是左 H -BiHom- 模余代数,

(c₃) (B, α_B, β_B) 是左 H -BiHom- 余模代数,

(c₄) $\Delta_B(ab) = a_1(a_{2(-1)} \triangleright \beta_B^{-1}(b_1)) \otimes \beta_B(a_{2(0)})b_2$,

$$(c_5) \alpha_H^{-1}((\alpha_H(h_1) \triangleright b)_{(-1)}) h_2 \otimes (\alpha_H(h_1) \triangleright b)_{(0)} = h_1 b_{(-1)} \otimes h_2 \triangleright b_{(0)},$$

其中 $a, b \in B, h \in H$. 这种情况下称 $(B, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 为 Radford 双积 monoidal BiHom- 双代数.

证明 ((i) \Rightarrow (ii)) 因为 $\epsilon_{B \times H}$ 是代数同态, 所以 $\epsilon_B(1_B) = \epsilon_{B \times H}(1_B \otimes 1_H) = 1$ 且

$$\epsilon_B(a(\beta_H^{-1}(h) \triangleright \beta_B^{-1}(b))) \epsilon_H(g) = \epsilon_B(a) \epsilon_B(b) \epsilon_H(h) \epsilon_H(g). \quad (1)$$

在(1)式中令 $h = g = 1_H$, 则 $\epsilon_B(ab) = \epsilon_B(a) \epsilon_B(b)$.

在(1)式中令 $a = 1_B, g = 1_H$, 则可得到 $\epsilon_B(h \triangleright b) = \epsilon_H(h) \epsilon_B(b)$.

因为 $\Delta_{B \times H}$ 是代数同态, 所以

$$1_B \otimes \beta_H(1_{B2(-1)}) \otimes \beta_B(1_{B2(0)}) \otimes 1_H = 1_B \otimes 1_H \otimes 1_B \otimes 1_H. \quad (2)$$

应用 $\epsilon_B \otimes Id \otimes Id \otimes \epsilon_H$ 作用于(2)式的两边, 可得到 $1_{B(-1)} \otimes 1_{B(0)} = 1_H \otimes 1_B$.

应用 $Id \otimes \epsilon_H \otimes Id \otimes \epsilon_H$ 作用于(2)式的两边, 可得到 $\Delta(1_B) = 1_B \otimes 1_B$.

由 $\Delta((a \otimes h)(b \otimes g)) = \Delta(a \otimes h) \Delta(b \otimes g)$ 得,

$$\begin{aligned} & (a(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)))_1 \otimes \alpha_H^{-1}(\beta_H((a(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)))_{2(-1)})) \beta_H^{-1}((\beta_H(h_2)g)_1) \otimes \\ & \beta_B((a(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)))_{2(0)}) \otimes (\beta_H(h_2)g)_2 = a_1(a_{2(-1)1} \alpha_H(\beta_H^2(h_{11})) \triangleright \beta_B^{-1}(b_1)) \otimes \\ & (\alpha_H^{-1}(\beta_H^2(a_{2(-1)2})h_{12})(\alpha_H^{-1}(\beta_H(b_{2(-1)})) \beta_H^{-1}(g_1))) \otimes \beta_B(a_{2(0)}) (\beta_H^{-1}(\alpha_H(h_{21})) \triangleright b_{2(0)}) \otimes \beta_H(h_{22})g_2. \end{aligned} \quad (3)$$

在(3)式中令 $h = g = 1_H$ 可得:

$$\begin{aligned} & (ab)_1 \otimes \beta_H((ab)_{2(-1)}) \otimes \beta_B((ab)_{2(0)}) \otimes 1_H = a_1(\alpha_H(a_{2(-1)1}) \triangleright \beta_B^{-1}(b_1)) \otimes \\ & \beta_H^2(a_{2(-1)2}) \beta_H(b_{2(-1)}) \otimes \beta_B(a_{2(0)}) \beta_B(b_{2(0)}) \otimes 1_H. \end{aligned} \quad (4)$$

用 $\epsilon_B \otimes Id \otimes Id \otimes \epsilon_H$ 作用于(4)式两端可得:

$$(ab)_{(-1)} \otimes (ab)_{(0)} = a_{(-1)} b_{(-1)} \otimes a_{(0)} b_{(0)}.$$

再用 $Id \otimes \epsilon_H \otimes Id \otimes \epsilon_H$ 作用于(4)式两端可得, 可得 (c_4) .

在(3)式中令 $a = 1_B, g = 1_H$, 有:

$$\begin{aligned} & (\alpha_H(h_1) \triangleright b)_1 \otimes \alpha_H^{-1}(\beta_H((\alpha_H(h_1) \triangleright b)_{2(-1)})) \beta_H^{-1}((\alpha_H(\beta_H(h_2)))_1) \otimes \\ & \beta_B((\alpha_H(h_1) \triangleright b)_{2(0)}) \otimes (\alpha_H(\beta_H(h_2)))_2 = (\alpha_H(h_{11}) \triangleright b_1) \otimes \\ & \beta_H(h_{12}) \beta_H(b_{2(-1)}) \otimes (\alpha_H(h_{21}) \triangleright \beta_B(b_{2(0)})) \otimes \alpha_H(\beta_H(h_{22})). \end{aligned} \quad (5)$$

用 $Id \otimes \epsilon_H \otimes Id \otimes \epsilon_H$ 作用于(5)式两端, 可得 $\Delta_B(h \triangleright b) = h_1 \triangleright b_1 \otimes h_2 \triangleright b_2$.

用 $\epsilon_B \otimes Id \otimes Id \otimes \epsilon_H$ 作用于(5)式两端, 可得 (c_5) .

因此条件 $(c_1) - (c_5)$ 成立.

((ii) \Rightarrow (i)) 易证 $\epsilon_{B \times H}$ 是代数同态和 $\Delta_{B \times H}(1_B \otimes 1_H) = 1_B \otimes 1_H \otimes 1_B \otimes 1_H$.

下证: $\Delta_{B \times H}((a \otimes h)(b \otimes g)) = \Delta_{B \times H}(a \otimes h) \Delta_{B \times H}(b \otimes g)$.

$$\begin{aligned} \Delta_{B \times H}((a \otimes h)(b \otimes g)) &= \Delta_{B \times H}(a(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)) \otimes \beta_H(h_2)g) = (a(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)))_1 \otimes \\ & \alpha_H^{-1}(\beta_H((\alpha_H(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)))_{2(-1)})) \beta_H^{-1}((\beta_H(h_2)g)_1) \otimes \beta_B((a(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)))_{2(0)}) \otimes \\ & (\beta_H(h_2)g)_2 = a_1(a_{2(-1)} \triangleright \beta_B^{-1}((\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b)))_1) \otimes \alpha_H^{-1}(\beta_H((\beta_B(a_{2(0)})(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \\ & \beta_B^{-1}(b))_{(-1)})) \beta_H^{-1}((\beta_H(h_2)g)_1) \otimes \beta_B((\beta_B(a_{2(0)})(\beta_H^{-1}(\alpha_H(h_1)) \triangleright \beta_B^{-1}(b))_{2(0)}) \otimes (\beta_H(h_2)g)_2 = \\ & a_1(a_{2(-1)} \triangleright \beta_B^{-1}((\beta_H^{-1}(\alpha_H(h_{11})) \triangleright \beta_B^{-1}(b_1)))) \otimes \alpha_H^{-1}(\beta_H((\beta_B(a_{2(0)})(\beta_H^{-1}(\alpha_H(h_{12})) \triangleright \\ & \beta_B^{-1}(b_2))_{(-1)})) (h_{21} \beta_H^{-1}(g_1)) \otimes \beta_B((\beta_B(a_{2(0)})(\beta_H^{-1}(\alpha_H(h_{12})) \triangleright \beta_B^{-1}(b_2))_{(0)}) \otimes \\ & \beta_H(h_{22})g_2 = a_1(a_{2(-1)} \triangleright \beta_B^{-1}((\beta_H^{-1}(\alpha_H(h_{11})) \triangleright \beta_B^{-1}(b_1)))) \otimes \alpha_H^{-1}(\beta_H((\beta_B(a_{2(0)})(\beta_H^{-1}(\alpha_H(h_{12})) \triangleright \\ & \beta_B^{-1}(b_2))_{(-1)})) (h_{21} \beta_H^{-1}(g_1)) \otimes \beta_B((\beta_B(a_{2(0)})(\beta_H^{-1}(\alpha_H(h_{12})) \triangleright \beta_B^{-1}(b_2))_{(0)}) \otimes \beta_H(h_{22})g_2 = \\ & a_1(a_{2(-1)} \triangleright \beta_B^{-1}((\beta_H^{-1}(\alpha_H(h_{11})) \triangleright \beta_B^{-1}(b_1)))) \otimes \alpha_H^{-1}(\beta_H^2(a_{2(0)}(h_{12}))) \\ & \alpha_H^{-1}(\beta_H((\beta_H^{-1}(\alpha_H^2(h_{121})) \triangleright \beta_B^{-1}(b_2))_{(-1)})) (\beta_H(h_{122}) \beta_H^{-1}(g_1)) \otimes \\ & \beta_B((\beta_B(a_{2(0)})(\beta_H^{-1}(\alpha_H(h_{121})) \triangleright \beta_B^{-1}(b_2))_{(0)}) \otimes h_2 g_2 = a_1(a_{2(-1)} \triangleright \end{aligned}$$

$$\begin{aligned}
 & \beta_B^{-1}((\beta_H^{-1}(\alpha_H(h_{11})) \triangleright \beta_B^{-1}(b_1))) \otimes (\alpha_H^{-1}(\beta_H^2(a_{2(0)(-1)}))(\alpha_H^{-2}(\beta_H((\beta_H^{-1} \cdot \\
 & (\alpha_H^2(h_{121})) \triangleright \beta_B^{-1}(b_2))_{(-1)})h_{122}))g_1 \otimes \beta_B^2(a_{2(0)(0)})\beta_B((\beta_H^{-1}(\alpha_H^2(h_{121})) \triangleright \\
 & \beta_B^{-1}(b_2))_{(0)}) \otimes h_2 g_2 = a_1(a_{2(-1)} \triangleright (\beta_H^{-2}(\alpha_H(h_{11})) \triangleright \beta_B^{-2}(b_1))) \otimes \\
 & (\alpha_H^{-1}(\beta_H^2(a_{2(0)(-1)}))(\alpha_H^{-2}(\beta_B(\beta_H^{-1}(\alpha_H^2(h_{121})) \triangleright \beta_B^{-1}(b_2))_{(-1)})h_{122}))g_1 \otimes \\
 & \beta_B^2(a_{2(0)(0)})\beta_B(\beta_H^{-1}(\alpha_H^2(h_{121})) \triangleright \beta_B^{-1}(b_2))_{(0)}) \otimes h_2 g_2 = a_1(a_{2(-1)} \triangleright \\
 & (\beta_H^{-2}(\alpha_H(h_{11})) \triangleright \beta_B^{-2}(b_1))) \otimes (\alpha_H^{-1}(\beta_H^2(a_{2(0)(-1)}))(\alpha_H^{-1}(\alpha_H^{-1}((\alpha_H^2(h_{121}) \triangleright \\
 & b_2)_{(-1)})\alpha_H(h_{122})))g_1 \otimes \beta_B^2(a_{2(0)(0)})\beta_B(\alpha_H^2(h_{121}) \triangleright b_2)_{(0)}) \otimes h_2 g_2 = \\
 & a_1(a_{2(-1)} \triangleright (\beta_H^{-2}(\alpha_H(h_{11})) \triangleright \beta_B^{-2}(b_1))) \otimes (\alpha_H^{-1}(\beta_H^2(a_{2(0)(-1)}))(\alpha_H^{-1}(\alpha_H(h_{121})b_{2(-1)})))g_1 \otimes \\
 & \beta_B^2(a_{2(0)(0)})\beta_B(\alpha_H(h_{122}) \triangleright b_{2(0)}) \otimes h_2 g_2 = a_1(\alpha_H(a_{2(-1)1}) \triangleright (\beta_H^{-2}(\alpha_H(h_{11})) \triangleright \\
 & \beta_B^{-2}(b_1))) \otimes (\alpha_H^{-1}(\beta_H^2(a_{2(-1)2}))(\alpha_H^{-1}(\alpha_H^{-1}(b_{2(-1)})))g_1 \otimes \beta_B(a_{2(0)})\beta_B(\alpha_H(h_{122}) \triangleright \\
 & b_{2(0)}) \otimes h_2 g_2 = a_1((a_{2(-1)1}\beta_H^{-2}(\alpha_H(h_{11})) \triangleright \beta_B^{-1}(b_1)) \otimes (\alpha_H^{-1}(\beta_H^2(a_{2(-1)2})) \cdot \\
 & (h_{121}\alpha_H^{-1}(b_{2(-1)})))g_1 \otimes \beta_B(a_{2(0)})\beta_B(\alpha_H(h_{122}) \triangleright b_{2(0)}) \otimes h_2 g_2 = \\
 & a_1((a_{2(-1)1}\beta_H^{-2}(\alpha_H(h_{11})) \triangleright \beta_B^{-1}(b_1)) \otimes (\alpha_H^{-1}(\beta_H^2(a_{2(-1)2}))(\alpha_H^{-1}(h_{12})\alpha_H^{-1}(b_{2(-1)})))g_1 \otimes \\
 & \beta_B(a_{2(0)})\beta_B(\alpha_H(\beta_H^{-1}(h_{21})) \triangleright b_{2(0)}) \otimes \beta_H(h_{22})g_2 = a_1((a_{2(-1)1}\beta_H^{-2}(\alpha_H(h_{11})) \triangleright \\
 & \beta_B^{-1}(b_1)) \otimes (\alpha_H^{-1}(\beta_H^2(a_{2(-1)2}))h_{12}) \cdot (\beta_H(\alpha_H^{-1}(b_{2(-1)}))\beta_H^{-1}(g_1))) \otimes \\
 & \beta_B(a_{2(0)})\beta_B(\alpha_H(\beta_H^{-1}(h_{21})) \triangleright b_{2(0)}) \otimes \beta_H(h_{22})g_2 = \Delta_{B \times H}(a \otimes h)\Delta_{B \times H}(b \otimes g).
 \end{aligned}$$

所以 $(B_{\#}^* H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 是 monoidal BiHom- 双代数.

证毕.

注 6 当 $\alpha_H = \beta_H, \alpha_B = \beta_B$ 时,此定理是文献[6]中的定理 3. 5.

命题 3 设 $(B_{\#}^* H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 是 Radford 双积 monoidal BiHom- 双代数. H 是带有对极 S_H 的 monoidal BiHom-Hopf 代数,在 $Hom(B, B)$ 中 $S_B: B \rightarrow B$ 是 Id 的卷积逆,且满足 $S_B \circ \beta_B = \beta_B \circ S_B$ 和 $S_B \circ \alpha_B = \alpha_B \circ S_B$,则 $(B_{\#}^* H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ 是 monoidal BiHom-Hopf 代数,对极 S 定义为 $S(b \otimes h) = (1_B \otimes S_H(\beta_H^{-1}(b_{(-1)})\alpha_H^{-2}(h)))(S_B(b_{(0)}) \otimes 1_H)$.

证明 对任意的 $b \in B$ 和 $h \in H$,有

$$\begin{aligned}
 (Id * S)(b \otimes h) &= (b_1 \otimes \alpha_H^{-1}(\beta_H(b_{2(-1)}))\beta_H^{-1}(h_1))S(\beta_B(b_{2(0)}) \otimes h_2) = \\
 & (b_1 \otimes \alpha_H^{-1}(\beta_H(b_{2(-1)}))\beta_H^{-1}(h_1))((1_B \otimes S_H(\beta_H^{-1}(\beta_B(b_{2(0)})_{(-1)})\alpha_H^{-2}(h_2))) \cdot \\
 (S_B(\beta_B(b_{2(0)})_{(0)}) \otimes 1_H) &= (b_1 \otimes \alpha_H^{-1}(\beta_H(b_{2(-1)}))\beta_H^{-1}(h_1))((1_B \otimes S_H(b_{2(0)(-1)}\alpha_H^{-2}(h_2))) \cdot \\
 (S_B(\beta_B(b_{2(0)(0)}) \otimes 1_H) &= ((\alpha_B^{-1}(b_1) \otimes \alpha_H^{-2}(\beta_H(b_{2(-1)}))\alpha_H^{-1}(\beta_H^{-1}(h_1)))(1_B \otimes \\
 S_H(b_{2(0)(-1)}\alpha_H^{-2}(h_2))) & (S_B(\beta_B^2(b_{2(0)(0)})) \otimes 1_H) = (b_1 \otimes (\beta_H(\alpha_H^{-2}(b_{2(-1)}))\alpha_H^{-1}(\beta_H^{-1}(h_1))) \cdot \\
 (S_H(b_{2(0)(-1)}\alpha_H^{-2}(h_2))) & (S_B(\beta_B^2(b_{2(0)(0)})) \otimes 1_H) = (b_1 \otimes (\beta_H(\alpha_H^{-2}(b_{2(-1)}))\alpha_H^{-1}(\beta_H^{-1}(h_1))) \cdot \\
 (S_H(\alpha_H^{-2}(h_2))S_H(b_{2(0)(-1)})) & (S_B(\beta_B^2(b_{2(0)(0)})) \otimes 1_H) = (b_1 \otimes (\beta_H(\alpha_H^{-2}(b_{2(-1)}))\alpha_H^{-2}(\beta_H^{-1}(h_1))) \cdot \\
 \beta_H^{-1}(S_H(\alpha_H^{-2}(h_2))) & S_H(\beta_H(b_{2(0)(-1)})))(S_B(\beta_B^2(b_{2(0)(0)})) \otimes 1_H) = (b_1 \otimes \beta_H(b_{2(-1)1})S_H \cdot \\
 (\beta_H(b_{2(-1)2})) & (S_B(\beta_B(b_{2(0)})) \otimes 1_H)\epsilon(h) = (b_1 \otimes 1_H)(S_B(b_2) \otimes 1_H)\epsilon(h) = \\
 (b_1 S_B(b_2) \otimes 1_H) & \epsilon(h) = (1_B \otimes 1_H)\epsilon(b \otimes h).
 \end{aligned}$$

相似的可证明 $(S * Id)(b \otimes h) = (1_B \otimes 1_H)\epsilon(b \otimes h)$. 证毕.

注 7 当 $\alpha_H = \beta_H, \alpha_B = \beta_B$ 时,此命题是文献[6]中的命题 3. 6.

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Radford Biproduct Monoidal BiHom-Hopf Algebra

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Abstract: Firstly, we introduce the notions of a smash product monoidal BiHom-algebra $(B \# H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ and a smash coproduct monoidal BiHom-coalgebra $(B \times H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$. Furthermore, we get sufficient and necessary conditions for $(B \# H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ and $(B \times H, \alpha_B \otimes \alpha_H, \beta_B \otimes \beta_H)$ to be a monoidal BiHom-bialgebra.

Keywords: Radford biproduct; Monoidal BiHom-Hopf algebra; Hopf algebra

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The Cauchy Problem for the Higher-Order Dispersive Equation

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Abstract: In this paper, we consider the Cauchy problem for the higher-order dispersive equation $u_t + \partial_x^{2j+1} u = \partial_x^{j+1} (u^2) + \partial_x^{-1} (u_x^2)$, $j \geq 2$, $j \in \mathbf{N}$, $x, t \in \mathbf{R}$. By using the modified Fourier restriction norm method and Strichartz estimate and the modified Bourgain space, we prove that the problem is locally well-posed in modified Sobolev space $H^{(s, \frac{1}{2j})}$ with $s > -\frac{j}{2} + \frac{3}{4}$. By using the iteration technique, we also prove that the flow map is not C^2 at the origin if we assume that the problem is well-posed in $H^{(s, w)}$ with $0 < w < \frac{1}{2j}$ for any $s \in \mathbf{R}$.

Keywords: Cauchy problem; local well-posedness; modified Sobolev spaces