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由微分算子和从属关系定义的解析函数类的包含关系

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摘 要:研究了三类单位圆盘内利用算子函数 $\mathcal{E}^{\lambda}_{\alpha,\beta}$ 定义的单叶解析函数类 $\mathcal{S}^{\lambda}_{\alpha,\beta}(\eta;\phi)$,$\mathcal{C}^{\lambda}_{\alpha,\beta}(\eta;\phi,\psi)$,$\mathcal{R}^{\lambda}_{\alpha,\beta}(\eta,\gamma;\phi,\psi)$,运用微分从属的理论研究得到了它们的包含关系,并结合 Nunokawa 引理得到其特殊子类的包含关系.

关键词:解析函数;微分算子;微分从属;Nunokawa 引理

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设 $\mathcal{A}$ 表示单位圆盘 $\boldsymbol{U}=\{z\in\mathbf{C}:|z|<1\}$ 内具有泰勒展开式 $f(z)=z+\sum\limits_{k=1}^{\infty}a_{k+1}z^{k+1}$ 的单叶解析函数族,g 在 $\boldsymbol{U}$ 内解析且由下式定义 $g(z)=z+\sum\limits_{k=1}^{\infty}b_{k+1}z^{k+1}$,$f$ 和 g 的 Hadamard 乘积(或卷积)定义为 $(f*g)(z)=z+\sum\limits_{k=1}^{\infty}a_{k+1}b_{k+1}z^{k+1}$.

称 f 从属于 g,记为 $f\prec g$ 或者 $f(z)\prec g(z)(z\in\boldsymbol{U})$,如果存在 $\boldsymbol{U}$ 内的 Schwarz 函数 $w(z)$(在 $\boldsymbol{U}$ 内解析且满足 $w(0)=0$,$|w(z)|<1$ 的函数)使得 $f(z)=g(w(z))$.特别地,如果 g 在 $\boldsymbol{U}$ 单叶,则 $f(z)\prec g(z)$ 等价于 $f(0)=g(0)$,$f(\boldsymbol{U})\subset(\boldsymbol{U})$.本文约定 $\mathcal{P}$ 表示所有的满足 $\mathcal{R}\{p(z)\}>0(z\in\boldsymbol{U})$,$p(0)=1$ 的正实部函数 $p(z)$ 组成的集合,$\mathcal{Q}$ 表示所有满足 $\phi(\boldsymbol{U})$ 为凸函数且关于实数轴对称的函数 $\phi(z)\in\mathcal{P}$ 组成的集合.若 $\phi\in\mathcal{Q}$,文献[1] 定义了函数类 $\mathcal{S}^*(\phi)$,$\mathcal{K}(\phi)$,$\mathcal{C}(\phi,\psi)$ 为:$\mathcal{S}^*(\phi)=\{f:f\in\mathcal{A},\dfrac{zf'(z)}{f(z)}\prec\phi(z),z\in\boldsymbol{U}\}$,$\mathcal{K}(\phi)=\{f:f\in\mathcal{A},1+\dfrac{zf''(z)}{f'(z)}\prec\phi(z),z\in\boldsymbol{U}\}$,和 $\mathcal{C}(\phi,\psi)=\{f:f\in\mathcal{A},g(z)\in\mathcal{S}^*(\psi),\dfrac{zf'(z)}{g(z)}\prec\phi(z),z\in\boldsymbol{U}\}$.

显然,$f(z)\in\mathcal{K}(\phi)$ 当且仅当 $zf'(z)\in\mathcal{S}^*(\phi)$.关于上述函数类及其推广,可以查阅文献[2-8].最近,文献[9] 引进了一类新奇的函数 $E^{\lambda}_{\alpha,\beta}(z)$ 如下:$E^{\lambda}_{\alpha,\beta}(z)=z+\sum\limits_{k=1}^{\infty}\dfrac{\Gamma(\beta)(\lambda)_k}{\Gamma(\alpha k+\beta)k!}z^{k+1}$,这里 $\alpha>0,\beta>0,\lambda>0$,其中 Γ 表示伽马函数,$(a)_k=a(a+1)\cdots(a+k-1)$.注意到,这类函数有下面很多种重要形式:$E^{1}_{1,1}(z)=ze^{z}$;$E^{2}_{1,1}(z)=z(z+1)e^{z}$;$E^{1}_{2,1}(z)=z\cosh(\sqrt{z})$;$E^{2}_{2,1}(z)=z\cosh(\sqrt{z})+\dfrac{1}{2}z\sqrt{z}\sinh(\sqrt{z})$;$E^{1}_{2,2}(z)=\sqrt{z}\sinh(\sqrt{z})$;$E^{2}_{2,2}(z)=\dfrac{1}{2}(\sqrt{z}\sinh(\sqrt{z})+z\cosh(\sqrt{z}))$;$E^{1}_{2,3}(z)=6(\dfrac{\sinh(\sqrt{z})}{\sqrt{z}}-1)$.利用函数 $E^{\lambda}_{\alpha,\beta}(z)$,结

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合 Hadamard 乘积,定义如下算子函数:$\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)=z+\sum_{k=1}^{\infty}\frac{\Gamma(\beta)(\lambda)_k}{\Gamma(\alpha k+\beta)k!}a_{k+1}z^{k+1}$,本文将上述参数进行推广为 α,β,λ 均为复数,以下同.有关算子函数 $\mathscr{E}_{\alpha,\beta}^{\lambda}$ 及其相应的性质,可以查阅文献[10—13].容易验算,算子函数具有下面的性质

$$\alpha z(\mathscr{E}_{\alpha,\beta+1}^{\lambda}f(z))'=\beta\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)+(\alpha-\beta)\mathscr{E}_{\alpha,\beta+1}^{\lambda}f(z),\tag{1}$$

$$z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'=(1-\lambda)\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)+\lambda\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z).\tag{2}$$

本文利用算子函数 $\mathscr{E}_{\alpha,\beta}$ 定义了 3 类函数类如下:

$$\mathscr{S}_{\alpha,\beta}^{\lambda}(\eta;\phi)=\left\{f:f\in\mathscr{A},\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}-\eta\right)\prec\phi(z),z\in\boldsymbol{U}\right\},\tag{3}$$

$$\mathscr{C}_{\alpha,\beta}^{\lambda}(\eta;\phi,\psi)=\left\{f:f\in\mathscr{A},\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)\in\mathscr{S}^*(\psi),\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}-\eta\right)\prec\phi(z),z\in\boldsymbol{U}\right\},\tag{4}$$

$$\mathscr{R}_{\alpha,\beta}^{\lambda}(\eta,\gamma;\phi,\psi)=\{f:f\in\mathscr{A},\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)\in\mathscr{S}^*(\psi),\frac{1}{1-\eta}((1-\gamma)\frac{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}+$$

$$\gamma\frac{(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'}-\eta)\prec\phi(z),z\in\boldsymbol{U}\}.\tag{5}$$

这里 $0\leq\eta<1,\gamma\geq0$,且 $\phi,\psi\in\mathscr{Q}$.特别地,当 $\eta=0$ 时,记 $\mathscr{S}_{\alpha,\beta}^{\lambda}(0;\phi)=\mathscr{S}_{\alpha,\beta}^{\lambda}(\phi)$. 注意到,对上述 3 类函数类中的参数选取合适的值,可以退化为一些特殊的函数类:

(1)取 $\alpha=0,\lambda=1$,则函数类 $\mathscr{S}_{0,\beta}^{1}(\eta;\phi)$ 由文献[14]定义并由文献[15]进行改进研究;

(2)取 $\alpha=0,\lambda=1,\eta=0$,则函数类 $\mathscr{S}_{0,\beta}^{1}(0;\phi)$ 就是函数类 $\mathscr{S}^*(\phi)$,$\mathscr{C}_{0,\beta}^{1}(0;\phi,\psi)$ 为近于凸函数.

当参数设置为其他形式时,文献[16—18]也做过相应的研究.综合上述文献的研究,本文主要利用从属性质研究上述 3 类函数类的包含关系及其特殊形式的函数类的包含关系.

1 主要结论

引理 1[19] 设函数 $h(z)$ 为 $\boldsymbol{U}$ 内的凸函数且满足 $\mathscr{R}\{\beta h(z)+\gamma\}>0$,函数 $p(z)$ 在 $\boldsymbol{U}$ 内解析且满足 $p(0)=h(0)=1$, 则从属关系 $p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h(z)\Rightarrow p(z)\prec h(z)$.

引理 2[19] 设函数 $h(z)$ 为 $\boldsymbol{U}$ 内的凸函数,$P:\boldsymbol{U}\to C$ 为 $\boldsymbol{U}$ 内的正实部函数,函数 $p(z)$ 在 $\boldsymbol{U}$ 内解析且满足 $p(0)=h(0)=1$, 则 $p(z)+P(z)\cdot zp'(z)\prec h(z)\Rightarrow p(z)\prec h(z)$.

定理 1 若 $\mathscr{R}\{\lambda\}\geq1-\eta,\mathscr{R}\{\frac{\beta}{\alpha}\}\geq1-\eta,\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)\neq0(z\in\boldsymbol{U}\backslash\{0\})$, 则

$$\mathscr{S}_{\alpha,\beta}^{\lambda+1}(\eta;\phi)\subset\mathscr{S}_{\alpha,\beta}^{\lambda}(\eta;\phi)\subset\mathscr{S}_{\alpha,\beta+1}^{\lambda}(\eta;\phi).$$

证明 先证明左边的包含关系.设 $f(z)\in\mathscr{S}_{\alpha,\beta}^{\lambda+1}(\eta;\phi)$, 由定义得

$$\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z)}-\eta\right)\prec\phi(z).\tag{6}$$

记

$$p(z)=\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}-\eta\right).\tag{7}$$

容易验证 $p(z)$ 在 $\boldsymbol{U}$ 内单叶且 $p(0)=1$. 由(2)式和(7)式,可以得到下面的等式关系

$$(1-\eta)p(z)+\eta+\lambda-1=\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}+\lambda-1=\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'+(\lambda-1)\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}=\frac{\lambda\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z)}{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}.\tag{8}$$

对(8)式两边取对数后再求导数,得

$$\frac{(1-\eta)p'(z)}{(1-\eta)p(z)+\eta+\lambda-1}=\frac{(\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z)}-\frac{(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}.\tag{9}$$

结合(7)式和(9)式得 $\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z)}=(1-\eta)p(z)+\eta+\frac{(1-\eta)zp'(z)}{(1-\eta)p(z)+\eta+\lambda-1}$, 因此

$$\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z)}-\eta\right)=p(z)+\frac{zp'(z)}{(1-\eta)p(z)+\eta+\lambda-1}.\tag{10}$$

由(6)式结合(10)式，得到下面的从属关系

$$p(z)+\frac{zp'(z)}{(1-\eta)p(z)+\eta+\lambda-1}\prec\phi(z).\tag{11}$$

由于 $\phi(z)\in\mathscr{Q}$ 为正实部函数，且 $\mathscr{R}\{(1-\eta)\phi(z)+\eta+\lambda-1\}>\mathscr{R}(\eta+\lambda-1)\geq 0$，因此，利用引理 1 和(11)式，得到 $p(z)\prec\phi(z)$，即 $f(z)\in\mathscr{S}_{\alpha,\beta}^{\lambda}(\eta;\phi)$，所以 $\mathscr{S}_{\alpha,\beta}^{\lambda+1}(\eta;\phi)\subset\mathscr{S}_{\alpha,\beta}^{\lambda}(\eta;\phi)$.

下面证明右边包含关系.设 $f(z)\in\mathscr{S}_{\alpha,\beta}^{\lambda}(\eta;\phi)$，则由(3)式，有

$$\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}-\eta\right)\prec\phi(z).\tag{12}$$

记

$$q(z)=\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}_{\alpha,\beta+1}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta+1}^{\lambda}f(z)}-\eta\right).\tag{13}$$

容易验证 $q(z)$ 在 $\boldsymbol{U}$ 单叶且 $q(0)=1$. 由(1)式和(13)式，有

$$(1-\eta)q(z)+\eta+\frac{\beta}{\alpha}-1=\frac{z(\mathscr{E}_{\alpha,\beta+1}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta+1}^{\lambda}f(z)}+\frac{\beta}{\alpha}-1=$$

$$\frac{z(\mathscr{E}_{\alpha,\beta+1}^{\lambda}f(z))'+(\frac{\beta}{\alpha}-1)\mathscr{E}_{\alpha,\beta+1}^{\lambda}f(z)}{\mathscr{E}_{\alpha,\beta+1}^{\lambda}f(z)}=\frac{\frac{\beta}{\alpha}\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}{\mathscr{E}_{\alpha,\beta+1}^{\lambda}f(z)}.\tag{14}$$

(14)式两端取对数后求导数，得

$$\frac{(1-\eta)q'(z)}{(1-\eta)q(z)+\eta+\frac{\beta}{\alpha}-1}=\frac{(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}-\frac{(\mathscr{E}_{\alpha,\beta+1}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta+1}^{\lambda}f(z)}.\tag{15}$$

结合(13)式和(15)式，得 $\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}=(1-\eta)q(z)+\eta+\frac{(1-\eta)zq'(z)}{(1-\eta)q(z)+\eta+\frac{\beta}{\alpha}-1}$. 因此

$$\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}-\eta\right)=q(z)+\frac{zq'(z)}{(1-\eta)q(z)+\eta+\frac{\beta}{\alpha}-1}.\tag{16}$$

再结合(12)、(16)式，得到

$$q(z)+\frac{zq'(z)}{(1-\eta)q(z)+\eta+\frac{\beta}{\alpha}-1}\prec\phi(z).\tag{17}$$

因为 $\phi(z)\in\mathscr{Q}$ 且 $\mathscr{R}\{(1-\eta)\phi(z)+\eta+\frac{\beta}{\alpha}-1\}>\mathscr{R}\{\eta+\frac{\beta}{\alpha}-1\}\geq 0$，利用引理 1 和(17)式，得到 $f(z)\in\mathscr{S}_{\alpha,\beta+1}^{\lambda}(\eta;\phi)$，即 $\mathscr{S}_{\alpha,\beta}^{\lambda}(\eta;\phi)\subset\mathscr{S}_{\alpha,\beta+1}^{\lambda}(\eta;\phi)$. 右边的包含关系得证.

由定理 1，很容易得到下面的结合，它将在后续的定理证明过程中用到.在定理 1 中令 $\eta=0$，有

$$\mathscr{S}_{\alpha,\beta}^{\lambda+1}(\phi)\subset\mathscr{S}_{\alpha,\beta}^{\lambda}(\phi).\tag{18}$$

定理 2　若 $\mathscr{R}\{\lambda\}\geq 1$，$\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)\neq 0\,(z\in\boldsymbol{U}\backslash\{0\})$，则 $\mathscr{C}_{\alpha,\beta}^{\lambda+1}(\eta;\phi,\psi)\subset\mathscr{C}_{\alpha,\beta}^{\lambda}(\eta;\phi,\psi)\subset\mathscr{C}_{\alpha,\beta+1}^{\lambda}(\eta;\phi,\psi)$.

证明　先证左边的包含关系.设 $f(z)\in\mathscr{C}_{\alpha,\beta}^{\lambda+1}(\eta;\phi,\psi)$，由(4)式，有下面的从属关系

$$\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda+1}g(z)}-\eta\right)\prec\phi(z),\tag{19}$$

这里 $\mathscr{E}_{\alpha,\beta}^{\lambda+1}g(z)\in\mathscr{S}^*(\psi)$，结合(18)式，有

$$\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda+1}g(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda+1}g(z)}\prec\psi(z)\Rightarrow\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}\prec\psi(z),\tag{20}$$

即 $\mathscr{E}_{\alpha,\beta}^{\lambda+1}g(z)\in\mathscr{S}^*(\psi)\Rightarrow\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)\in\mathscr{S}^*(\psi)$，现记

$$u(z)=\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}-\eta\right). \tag{21}$$

容易验证 $u(z)$ 在 $\boldsymbol{U}$ 内解析且 $u(0)=1$. 将(21)式做恒等变形,得

$$[(1-\eta)u(z)+\eta]\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)=z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'. \tag{22}$$

由于 $z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'=\mathscr{E}_{\alpha,\beta}^{\lambda}(zf'(z))$,对(22)式两端求导数,得 $(1-\eta)u'(z)\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)+[(1-\eta)u(z)+\eta](\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'=(\mathscr{E}_{\alpha,\beta}^{\lambda}(zf'(z)))'$,即

$$(1-\eta)zu'(z)+[(1-\eta)u(z)+\eta]\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}=\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}(zf'(z)))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}. \tag{23}$$

结合(2)、(21)、(23)式,经过推导

$$\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda+1}g(z)}=\frac{\lambda\mathscr{E}_{\alpha,\beta}^{\lambda+1}(zf'(z))}{\lambda\mathscr{E}_{\alpha,\beta}^{\lambda+1}g(z)}=\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}(zf'(z)))'+(\lambda-1)\mathscr{E}_{\alpha,\beta}^{\lambda}(zf'(z))}{z(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'+(\lambda-1)\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}=$$

$$\frac{\dfrac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}(zf'(z)))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}+\dfrac{(\lambda-1)\mathscr{E}_{\alpha,\beta}^{\lambda}(zf'(z))}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}}{\dfrac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}+(\lambda-1)}=(1-\eta)\frac{zu'(z)}{\dfrac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}+\lambda-1}+(1-\eta)u(z)+\eta,$$

所以

$$\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda+1}g(z)}-\eta\right)=u(z)+\frac{zu'(z)}{Q(z)+\lambda-1}, \tag{24}$$

这里 $Q(z)=\dfrac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}\prec\psi(z)(\psi(z)\in\mathscr{Q})$. 由于

$$\mathscr{R}\left\{\frac{1}{Q(z)+\lambda-1}\right\}=\mathscr{R}\left\{\frac{\overline{Q(z)+\lambda-1}}{(Q(z)+\lambda-1)(\overline{Q(z)+\lambda-1})}\right\}=$$

$$\frac{1}{|Q(z)+\lambda-1|^2}\mathscr{R}\{Q(z)+\lambda-1\}>\frac{\mathscr{R}\{\lambda-1\}}{|Q(z)+\lambda-1|^2}\geq 0,$$

利用引理 2 和(19)、(24)式,得到 $u(z)\prec\phi(z)$,即 $f(z)\in\mathscr{C}_{\alpha,\beta}^{\lambda}(\eta;\phi,\psi)$,即 $\mathscr{C}_{\alpha,\beta}^{\lambda+1}(\eta;\phi,\psi)\subset\mathscr{C}_{\alpha,\beta}^{\lambda}(\eta;\phi,\psi)$,结论的第一部分成立.证明的第二部分同定理 1,这里由于篇幅原因故省掉.

定理 3 若 $\gamma\geq 0$,$\mathscr{N}_{\xi}^{\lambda}(a,c)g(z)\neq 0(z\in\boldsymbol{U}\backslash\{0\})$,则 $\mathscr{R}_{\alpha,\beta}^{\lambda}(\eta,\gamma;\phi,\psi)\subset\mathscr{R}_{\alpha,\beta}^{\lambda}(\eta,0;\phi,\psi)$.

证明 当 $\gamma=0$ 时,结论显然成立,下面讨论 $\gamma>0$ 的情形.设 $f(z)\in\mathscr{R}_{\alpha,\beta}^{\lambda}(\eta,\gamma;\phi,\psi)$,由(5)式,有从属关系

$$\frac{1}{1-\eta}\left((1-\gamma)\frac{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}+\gamma\frac{(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'}-\eta\right)\prec\phi(z), \tag{25}$$

这里 $\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)\in\mathscr{S}^*(\psi)$,即 $\dfrac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}\prec\psi(z)(\psi(z)\in\mathscr{Q})$. 记

$$s(z)=\frac{1}{1-\eta}\left(\frac{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}-\eta\right). \tag{26}$$

容易验证 $s(z)$ 在 $\boldsymbol{U}$ 内单叶且 $s(0)=1$. 由(26)式,有

$$[(1-\eta)s(z)+\eta]\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)=\mathscr{E}_{\alpha,\beta}^{\lambda}f(z). \tag{27}$$

对(27)式两端求导数,得 $(1-\eta)s'(z)\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)+[(1-\eta)s(z)+\eta](\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'=(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'$,即

$$(1-\eta)s(z)+\eta+\frac{(1-\eta)s'(z)\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}{(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'}=\frac{(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'}. \tag{28}$$

结合(27)、(28)式,有

$$(1-\gamma)\frac{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z)}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}+\gamma\frac{(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z))'}{(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'}=(1-\eta)s(z)+\eta+(1-\eta)\gamma\frac{zs'(z)}{P(z)}, \tag{29}$$

这里 $P(z)=\dfrac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}g(z))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)}\prec\psi(z)$,由从属的定义和 $\psi(z)\in\mathscr{Q}$,得 $\mathscr{R}\{P(z)\}>0$. 结合(25)、(29)式,有

$$s(z)+\gamma\frac{zs'(z)}{P(z)}\prec\phi(z).\tag{30}$$

而 $\mathscr{R}\left\{\frac{\gamma}{P(z)}\right\}=\mathscr{R}\left\{\frac{\gamma\,\overline{P(z)}}{P(z)\,\overline{P(z)}}\right\}=\frac{\gamma}{|P(z)|^2}\mathscr{R}\{P(z)\}>0$,利用引理 2 和(30)式,得 $s(z)\prec\phi(z)$,即 $f(z)\in\mathscr{R}^{\lambda}_{\alpha,\beta}(\eta,0;\phi,\psi)$. 定理 3 得证.

下面取 $0<\alpha\leq 1,\phi(z)=(\frac{1+z}{1-z})^{\alpha}$,上述 3 类函数类将退化为幅角函数类

$$\mathscr{SS}^{\lambda}_{\alpha,\beta}(\eta,\alpha)=\left\{f: f\in\mathscr{A},\left|\arg\left(\frac{z(\mathscr{E}^{\lambda}_{\alpha,\beta}f(z))'}{\mathscr{E}^{\lambda}_{\alpha,\beta}f(z)}-\eta\right)\right|<\frac{\pi\alpha}{2},z\in\boldsymbol{U}\right\},\tag{31}$$

$$\mathscr{SC}^{\lambda}_{\alpha,\beta}(\eta,\alpha,\psi)=\left\{f: f\in\mathscr{A},\mathscr{E}^{\lambda}_{\alpha,\beta}g(z)\in\mathscr{S}^*(\psi),\left|\arg\left(\frac{z(\mathscr{E}^{\lambda}_{\alpha,\beta}f(z))'}{\mathscr{N}^{\lambda}_{\xi}(a,c)g(z)}-\eta\right)\right|<\frac{\pi\alpha}{2},z\in\boldsymbol{U}\right\},\text{和}$$

$$\mathscr{SR}^{\lambda}_{\alpha,\beta}(\eta,\gamma,\alpha,\psi)=\{f: f\in\mathscr{A},\mathscr{E}^{\lambda}_{\alpha,\beta}g(z)\in\mathscr{S}^*(\psi),|\arg((1-\gamma)\frac{\mathscr{E}^{\lambda}_{\alpha,\beta}f(z)}{\mathscr{E}^{\lambda}_{\alpha,\beta}g(z)}+$$
$$\gamma\frac{(\mathscr{E}^{\lambda}_{\alpha,\beta}f(z))'}{(\mathscr{E}^{\lambda}_{\alpha,\beta}g(z))'}-\eta)|<\frac{\pi\alpha}{2},z\in\boldsymbol{U}\}.$$

下面得到这几类特殊函数类的一些性质.

引理 3(Nunokawa 引理[20])　设函数 $p(z)$ 在 $\boldsymbol{U}$ 内解析,$p(z)\neq 0$,且定义为 $p(z)=1+\sum\limits_{k=m}^{\infty}p_k z^k(p_m\neq 0)$. 若存在某个正数 $0<\alpha\leq 1$ 和 $\boldsymbol{U}$ 内某点 $z_0(|z_0|<1)$,使得在 $|z|<|z_0|$ 内有 $|\arg\{p(z)\}|<\frac{\pi\alpha}{2}(|z|<|z_0|)$,在点 z_0 处有 $|\arg\{p(z_0)\}|=\frac{\pi\alpha}{2}$,则存在实数 $l\geq\frac{m(a+a^{-1})}{2}\geq m$,使得 $\frac{z_0p'(z_0)}{p(z_0)}=\frac{2il\arg\{p(z_0)\}}{\pi}$,这里 $[p(z_0)]^{1/\alpha}=\pm ia(a>0)$.

定理 4　若 $\mathscr{R}\{\lambda\}\geq 1-\eta,\mathscr{R}\{\frac{\beta}{\alpha}\}\geq 1-\eta,\mathscr{E}^{\lambda}_{\alpha,\beta}f(z)\neq 0(z\in\boldsymbol{U}\backslash\{0\})$,则存在 $r>0,m\in\mathbf{N}$ 和 $0\leq\delta<\alpha$,使得 $\mathscr{SS}^{\lambda+1}_{\alpha,\beta}(\eta,\sigma)\subset\mathscr{SS}^{\lambda}_{\alpha,\beta}(\eta,\alpha)$,这里

$$\sigma=\alpha+\frac{2}{\pi}\arctan[m\alpha\cos(\frac{\pi\delta}{2})/(r+m\alpha\sin(\frac{\pi\delta}{2}))].\tag{32}$$

证明　设 $f(z)\in\mathscr{SS}^{\lambda+1}_{\alpha,\beta}(\eta,\sigma)$,由(31)式,有

$$\left|\arg\left(\frac{z(\mathscr{E}^{\lambda+1}_{\alpha,\beta}f(z))'}{\mathscr{E}^{\lambda+1}_{\alpha,\beta}f(z)}-\eta\right)\right|<\frac{\pi\sigma}{2}(0\leq\eta<1;0<\sigma\leq 1),\tag{33}$$

存在函数 $\phi(z)$ 定义为 $\phi(z)=(\frac{1+z}{1-z})^{\sigma}\in\mathscr{Q}$,使得

$$\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}^{\lambda+1}_{\alpha,\beta}f(z))'}{\mathscr{E}^{\lambda+1}_{\alpha,\beta}f(z)}-\eta\right)\prec\phi(z).\tag{34}$$

定义

$$p(z)=\frac{1}{1-\eta}\left(\frac{z(\mathscr{E}^{\lambda}_{\alpha,\beta}f(z))'}{\mathscr{E}^{\lambda}_{\alpha,\beta}f(z)}-\eta\right).\tag{35}$$

由定理 1 的分析和(34)、(35)式,有 $p(z)\prec\phi(z)$.由于 $\phi(z)\in\mathscr{Q}$,且 $p(z)\neq 0$,因此 $\arg\{p(z)\}$ 有意义.

下面利用反证法证明定理 4 的结论.假设存在一点 $z_0(|z_0|<1)$ 和实数 $0<\alpha\leq 1$ 使得在 $|z|<|z_0|$ 内有 $|\arg\{p(z)\}|<\frac{\pi\alpha}{2}(|z|<|z_0|)$,在 z_0 处有 $|\arg\{p(z_0)\}|=\frac{\pi\alpha}{2}$,利用引理 4,存在实数 l 满足 $l\geq\frac{m(a+a^{-1})}{2}\geq m$,使得

$$\frac{z_0p'(z_0)}{p(z_0)}=\frac{2il\arg\{p(z_0)\}}{\pi},\tag{36}$$

这里 $[p(z_0)]^{1/\alpha}=\pm \mathrm{i}a(a>0)$. 由(10)式,有

$$\arg\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z_0))'}{\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z_0)}-\eta\right)=\arg\left(p(z_0)+\frac{z_0p'(z_0)}{(1-\eta)p(z_0)+\eta+\lambda-1}\right)=$$
$$\arg\left(p(z_0)(1+\frac{z_0p'(z_0)}{p(z_0)[(1-\eta)p(z_0)+\eta+\lambda-1]})\right)=\arg\{p(z_0)\}+$$
$$\arg\left(1+\frac{z_0p'(z_0)}{p(z_0)[(1-\eta)p(z_0)+\eta+\lambda-1]}\right). \tag{37}$$

引入复数的模 $r(r>0)$ 和幅角 $\delta(0\leq\delta<\alpha)$,使得

$$(1-\eta)p(z_0)+\eta+\lambda-1=\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda}f(z_0))'}{\mathscr{E}_{\alpha,\beta}^{\lambda}f(z_0)}+\lambda-1\doteq r\mathrm{e}^{\pm\mathrm{i}\frac{\pi\delta}{2}}.$$

如果在点 z_0 处有 $\arg\{p(z_0)\}=\frac{\pi\alpha}{2}$,则由(36)式有 $\arg\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z_0))'}{\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z_0)}-\eta\right)=\frac{\pi\alpha}{2}+\arg(1+\frac{\mathrm{i}l\alpha}{r}\mathrm{e}^{\mp\mathrm{i}\frac{\pi\delta}{2}})\geq$ $\frac{\pi\alpha}{2}+\arctan[l\alpha\cos(\frac{\pi\delta}{2})/(r+l\alpha\sin(\frac{\pi\alpha}{2}))]$. 容易验证,上式最右端的函数 $K(l)=l\alpha\cos(\frac{\pi\delta}{2})/(r+l\alpha\sin(\frac{\pi\alpha}{2}))$ 单调递增,因此由(32)式,有

$$\arg p(z_0)=\arg\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z_0))'}{\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z_0)}-\eta\right)\geq\frac{\pi\alpha}{2}+\arctan[m\alpha\cos(\frac{\pi\delta}{2})/(r+m\alpha\sin(\frac{\pi\alpha}{2}))]=\frac{\pi\sigma}{2}.$$

如果在点 z_0 处有 $\arg\{p(z_0)\}=-\frac{\pi\alpha}{2}$,利用相同的方法,有

$$\arg p(z_0)=\arg\left(\frac{z(\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z_0))'}{\mathscr{E}_{\alpha,\beta}^{\lambda+1}f(z_0)}-\eta\right)=-\frac{\pi\alpha}{2}+\arg(1-\frac{\mathrm{i}l\alpha}{r}e^{\mp\mathrm{i}\frac{\pi\delta}{2}})\leq$$
$$-\frac{\pi\alpha}{2}-\arctan[m\alpha\cos(\frac{\pi\delta}{2})/(r+m\alpha\sin(\frac{\pi\alpha}{2}))]=-\frac{\pi\sigma}{2}.$$

综合可得 $|\arg\{p(z_0)\}|\geq\frac{\pi\sigma}{2}$,这与条件(33)式产生矛盾,因此在 $\boldsymbol{U}$ 内不存在点 z_0 满足 $|\arg\{p(z_0)\}|=\frac{\pi\alpha}{2}$,即对所有 $z\in\boldsymbol{U}$ 都有 $|\arg\{p(z)\}|<\frac{\pi\alpha}{2}$,即 $f(z)\in\mathscr{SS}_{\alpha,\beta}^{\lambda}(\eta,\alpha)$. 定理 4 得证.

利用同样的方法,可以得到定理 5 和定理 6.

定理 5 若 $\mathscr{R}\{\lambda\}\geq 1$,$\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)\neq 0(z\in\boldsymbol{U}\backslash\{0\})$,则存在 $r_1>0$,$m\in\mathbf{N}$ 和 $0\leq\delta_1<\alpha$ 使得 $\mathscr{SC}_{\alpha,\beta}^{\lambda+1}(\eta,\sigma_1,\psi)\subset\mathscr{SC}_{\alpha,\beta}^{\lambda}(\eta,\alpha,\psi)$, 这里

$$\sigma_1=\alpha+\frac{2}{\pi}\arctan[m\alpha\cos(\frac{\pi\delta_1}{2})/(r_1+m\alpha\sin(\frac{\pi\delta_1}{2}))].$$

定理 6 若 $\gamma\geq 0$,$\mathscr{E}_{\alpha,\beta}^{\lambda}g(z)\neq 0(z\in\boldsymbol{U}\backslash\{0\})$,则存在 $r_2>0$,$m\in\mathbf{N}$ 和 $0\leq\delta_2<\alpha$ 使得 $\mathscr{SR}_{\alpha,\beta}^{\lambda}(\eta,\gamma,\sigma_2,\psi)\subset\mathscr{SR}_{\alpha,\beta}^{\lambda}(\eta,0,\alpha,\psi)$, 这里 $\sigma_2=\alpha+\frac{2}{\pi}\arctan(m\alpha)\cos(\frac{\pi\delta_2}{2})/(r_2+m\alpha\sin(\frac{\pi\delta_2}{2}))$.

2 结束语

本文主要研究函数 $E_{\alpha,\beta}^{\lambda}(z)$[9] 结合 Hadamard 乘积引入的算子函数,定义了三类单位圆盘上的单叶解析函数类,该函数类是近于凸函数、星象函数类的推广,利用从属关系的理论,研究得到了它们的包含关系.从函数类的定义来看,不仅仅单叶函数具有此类性质,也可以将该方法和理论推广到多叶函数.对于多叶函数的简单形式,文献[21-23]进行过相应的研究.

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Inclusion relationships for certain subclasses of analytic functions involving differntial operator and subordination

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Abstract：In this paper，by using the operator $\mathscr{E}_{\alpha,\beta}^{\lambda}$ we define three subclasses of analytic functions $\mathscr{S}_{\alpha,\beta}^{\lambda}(\eta;\phi)$，$\mathscr{C}_{\alpha,\beta}^{\lambda}(\eta;\phi,\psi)$，$\mathscr{R}_{\alpha,\beta}^{\lambda}(\eta,\gamma;\phi,\psi)$ in the unit disc.The object of this paper is to investigate the inclusion relationship of these subclasses with the theory of differential subordination .Combination with Nunokawa lemma，we also obtain the inclusion relationship of special subclasses.

Keywords：analytic function；differential operator；differential subordination；nunokawa lemma

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