

# 变系数模型 CVaR 约束下最优投资组合选择

## ——基于非广延统计理论

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**摘要:**考虑到经济变量随时间的变化而变化,设想资产的收益和协方差矩阵为时间的函数是合理的.一些实证结果表明,风险资产的收益分布呈现出厚尾特征.因此,应用 Tsallis 分布来驱动风险资产收益的变化,可以更好地捕捉厚尾这一特征.基于非广延统计理论,在 CVaR 约束下,提出了连续时间最优投资组合选择问题.针对模型中的时变系数,采用局部常数拟合的方法.首先,在非广延统计理论下,构造了风险资产的价格过程.然后,利用随机动态规划方法得到了 HJB 方程.在对数效用函数下,利用拉格朗日乘子法,得到了对数效用函数下具有 CVaR 约束的最优投资组合策略,并通过实证分析展示了所得结论的拟合效果.

**关键词:**非广延统计理论;投资组合选择;CVaR 约束;时变模型;HJB 方程

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投资组合选择问题是金融领域中非常重要的核心问题之一.文献[1]首次引入了单周期均值一方差方法求解最优投资组合问题.文献[2]将 Markowitz 的模型扩展到连续时间均值一方差组合选择模型.文献[3]研究了多周期投资组合选择中均值一方差公式的最优解析解,提出了一种最优投资组合策略的有效算法.文献[4]利用拉格朗日对偶方法和动态规划方法研究了多资产风险资产连续时间均值一方差投资组合选择问题.文献[5]研究了收益参数向量的估计问题.然而,在金融市场中,投资者往往会考虑一些交易限制,以进行有效的风险管理.例如,文献[6]提出不允许卖空下最优消费和投资的对偶方法.文献[7]在较高借贷利率约束下,得到动态均值一方差投资组合选择问题最优投资策略的显式闭式解.文献[8]利用 VaR 分析了最优动态投资组合.文献[9]研究了条件 VaR 约束下的消费投资组合问题.

然而,在上述模型中,资产的价格过程大都由经典布朗运动驱动,其收益的分布为正态分布.现有文献的实证结果表明,股票收益分布往往为正偏态分布,资产收益呈现厚尾特征,如文献[10-11]等.为解决这一问题,很多研究者做了大量的工作.1988 年, Tsallis 引入了非广延统计理论,非广延统计理论是经典 Boltzmann-Gibbs<sup>[12]</sup>统计的推广.文献[13]将非广延统计方法应用于金融市场,来描述厚尾特征.文献[14-15]分别将非广延统计理论应用于 VaR 约束和流动性约束下的投资组合选择问题.文献[16]研究了用非广延统计理论对资产价格过程建模的最优投资问题.文献[17]讨论了非广延统计理论下具有非线性财富方程的连续时间均值一方差组合选择问题.文献[18]研究了重尾分布下延迟索赔风险过程的精细大偏差.这些研究结果表明,非广延统计方法能很好地适用于金融模型.

本文基于非广延统计理论,主要研究时变系数模型在 CVaR 约束下连续时间最优投资组合选择问题.利用动态规划原理得到 HJB 方程,并利用拉格朗日乘子法给出最优策略的显式闭式解.

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## 1 市场模型与 CVaR 约束

考虑到风险资产收益率的分布呈现厚尾特征, 本文将非广延统计理论应用于风险资产的价格过程. 假设  $S(t)$  为在时刻  $t$  的风险资产价格过程. 在滤子概率空间  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$  上,  $S(t)$  满足如下随机微分方程

$$dS(t) = S(t)(\mu(t)dt + \sigma(t)d\Omega(t)), \quad (1)$$

其中  $d\Omega(t) = P(\Omega, t)^{\frac{1-q}{2}} dB(t)$ ,  $B(t)$  是标准布朗运动,  $P(\Omega, t)$  是参数为  $q$  的 Tsallis 分布, 即:

$$P(\Omega, t) = \frac{1}{z(t)}(1 - \beta(t)(1 - q)\Omega^2)^{\frac{1}{1-q}}, \quad (2)$$

其中  $z(t) = [(2 - q)(3 - q)ct]^{\frac{1}{3-q}}$ ,  $\beta(t) = c^{\frac{1-q}{3-q}}[(2 - q)(3 - q)t]^{\frac{-2}{3-q}}$ ,  $c = \frac{\pi}{q-1} \cdot \frac{\Gamma^2(\frac{1}{1-q} - \frac{1}{2})}{\Gamma^2(\frac{1}{q-1})}$ , 特别地, 当

$q \rightarrow 1$  时, 模型(1)服从时变的 Black-Scholes 模型. BORLAND<sup>[19]</sup>的结果表明, 当  $1 < q < \frac{5}{3}$  时, Tsallis 分布呈

现幂律尾部, 并具有有限方差. 因此, 该模型是经典的 Black-Scholes 模型的推广.

假设金融市场存在  $m+1$  种资产, 一种是无风险债券, 其价格过程  $S_0(t)$  满足如下的微分方程:

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, t \in [0, T], \\ S_0(0) = s_0 > 0, \end{cases}$$

其中  $r(t)$  是  $t$  时刻的无风险利率,  $T$  为投资的终止时间. 另  $m$  种资产是股票, 其价格过程满足如下随机微分方程:

$$\begin{cases} dS_i(t) = S_i(t)[\mu_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)d\Omega_j(t)], \\ S_i(0) = s_i > 0, i = 1, 2, \dots, m \quad t \in [0, T], \end{cases}$$

其中  $d\Omega_j(t) = P(\Omega_j, t)^{\frac{1-q_j}{2}} dB_j(t)$ ,  $j = 1, 2, \dots, m$ , 这里  $B_j(t)$ ,  $j = 1, 2, \dots, m$  是标准布朗运动,  $P(\Omega_j, t)$  是(2)定义的参数为  $q$  的 Tsallis 分布. 记收益率矩阵  $\mu(t) = (\mu_1(t), \mu_2(t), \dots, \mu_m(t))^T$ , 波动率矩阵  $\sigma(t) = (\sigma_{ij}(t), i, j = 1, 2, \dots, m)$ . 令  $L_{\mathcal{F}}^2([0, T]; \mathbf{R}^m)$  表示所有  $(\mathcal{F}_t, t \geq 0)$  适应的, 在  $\mathbf{R}^m$  上取值且平方可积的随机过程的全体. 设  $\pi(t) = (\pi_1(t), \pi_2(t), \dots, \pi_m(t)) \in L_{\mathcal{F}}^2([0, T]; \mathbf{R}^m)$ , 其分量  $\pi_i(t)$ ,  $(i = 1, 2, \dots, m)$  为投资者  $t$  时刻的财富在第  $i$  个风险资产投资的比例.  $c(t)$  为  $t$  时刻投资者的消费. 一个控制策略  $\{\pi(t), c(t)\}$  为适应的, 如果  $\{\pi(t), c(t)\} \in L_{\mathcal{F}}^2([0, T]; \mathbf{R}^{m+1})$  且关于  $\mathcal{F}_t$  递增可测.

令  $X^{\pi, c}(t)$  为投资者的财富过程, 则财富过程满足如下的随机微分方程:

$$dX^{\pi, c}(t) = X^{\pi, c}(t)[(\pi^T(t)\mu(t) - c(t))dt + \pi^T(t)\sigma(t)d\Omega(t)] + X^{\pi, c}(t)r(t)[1 - \sum_{i=1}^m \pi_i(t)]dt = X^{\pi, c}(t)[(\pi^T(t)\mu^*(t) - c(t) + r(t))dt + \pi^T(t)\sigma(t)P_q dB(t)], \quad (3)$$

其中:  $\mu^*(t) = (\mu_1(t) - r(t), \mu_2(t) - r(t), \dots, \mu_m(t) - r(t))^T$ ,  $P_q = \text{diag}(P(\Omega_1, t)^{\frac{1-q_1}{2}}, P(\Omega_2, t)^{\frac{1-q_2}{2}}, \dots, P(\Omega_m, t)^{\frac{1-q_m}{2}})$ , 这里  $\text{diag}(x)$  表示以向量  $x$  为对角线元素的对角阵. 投资组合策略过程  $\{\pi(t), c(t)\}$  为适应过程而且满足以下条件:

$$\int_0^t |\pi^T(s)\mu^*(s)| ds + \int_0^t \|\pi^T(s)\sigma(s)P_q\| ds + \int_0^t c(s)ds < \infty, \text{ a.s.},$$

其中  $\|\cdot\|$  表示欧几里得范数. 令

$$H(t, \pi(t), c(t)) = \pi^T(t)\mu^*(t) + r(t) - c(t) - \frac{1}{2} \|\pi^T(t)\sigma(t)P_q\|^2.$$

由伊藤公式, (3)的唯一解由下式给出:

$$X^{\pi, c}(t) = X^{\pi, c}(0) \exp\left\{\int_0^t H(s, \pi(s), c(s))ds + \int_0^t \pi^T(s)\sigma(s)P_q dB(s)\right\},$$

其中  $X^{\pi,c}(0) = X(0)$  为投资者的初始财富,对于一个固定的时间点  $t$  和足够小的  $\tau > 0$ ,由局部拟合方法(见文献[19]),则对于  $s \in [t, t + \tau]$ ,有  $r(s) = r(t), \mu(s) = \mu(t), \sigma(s) = \sigma(t), \pi(s) = \pi(t)$  和  $c(s) = c(t)$ .这种局部拟合方法在金融市场是合理的,事实上,对于充分短的时间间隔,控制策略  $\{\pi(t), c(t)\}$ ,收益率和波动率不变.因此,在这些假设下,  $t + \tau$  时刻的财富可以表示为

$$X^{\pi,c}(t + \tau) = X^{\pi,c}(t) \exp\{H(t, \pi(t), c(t))\tau + \pi^T(t)\sigma(t)P_q(B(t + \tau) - B(t))\},$$

因此在时间区间  $[t, t + \tau]$  上的财富损失函数为

$$L^{\pi,c}(t) = X^{\pi,c}(t) - X^{\pi,c}(t + \tau) = X^{\pi,c}(t)[1 - \exp\{H(t, \pi(t), c(t))\tau + \pi^T(t)\sigma(t)P_q(B(t + \tau) - B(t))\}], \tag{4}$$

对于固定时间点  $t$ ,随机变量  $\pi^T(t)\sigma(t)P_q(B(t + \tau) - B(t))$  服从均值为零,标准差为  $\|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau}$  的正态分布.取置信水平  $\alpha \in (0.5, 1]$ ,在实际风险管理中  $\alpha$  通常取值为 0.95 或 0.99,由定义  $\alpha = P(L^{\pi,c}(t) \leq VaR_\alpha)$  和  $CVaR_\alpha(L^{\pi,c}(t)) = E[L^{\pi,c}(t) \geq VaR_\alpha]$ ,给出下面的定理 1.

**定理 1** 设  $L^{\pi,c}(t)$  为(4)式中在时间区间  $[t, t + \tau]$  上的财富损失,置信水平  $\alpha \in (0.5, 1]$ ,则可得

$$VaR_\alpha = X^{\pi,c}(t)[1 - \exp\{H(t, \pi(t), c(t))\tau + \|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau} b_1\}],$$

$$CVaR_\alpha(L^{\pi,c}(t)) = X^{\pi,c}(t)[1 - b_2(t) \exp\{H(t, \pi(t), c(t))\tau + \frac{1}{2} \|\pi^T(t)\sigma(t)P_q\|^2 \tau\}],$$

其中  $b_2(t) = \Phi(b_1 - \|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau}) / (1 - \alpha), b_1 = \Phi^{-1}(1 - \alpha)$ . 这里的  $\Phi$  和  $\Phi^{-1}$  分别为标准正态分布的分布函数和反函数.

**证明** 损失  $L^{\pi,c}(t)$  的分布函数为

$$F_L(x) = P\{L^{\pi,c}(t) \leq x\} = P\{X^{\pi,c}(t)[1 - \exp\{H(t, \pi(t), c(t))\tau + \pi^T(t)\sigma(t)P_q \Delta B_t\}] \leq x\} =$$

$$P\left\{\frac{1}{\|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau}} \ln \frac{X^{\pi,c}(t) - x}{X^{\pi,c}(t) \exp\{H(t, \pi(t), c(t))\tau\}} \leq \frac{\Delta B_t}{\sqrt{\tau}}\right\} =$$

$$1 - \Phi\left(\frac{1}{\|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau}} \ln \frac{X^{\pi,c}(t) - x}{X^{\pi,c}(t) \exp\{H(t, \pi(t), c(t))\tau\}}\right).$$

因此,损失的概率密度函数为

$$f_L(x) = F'_L(x) = \phi\left(\frac{1}{\|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau}} \ln \frac{X^{\pi,c}(t) - x}{X^{\pi,c}(t) \exp\{H(t, \pi(t), c(t))\tau\}}\right) \cdot$$

$$\frac{1}{(X^{\pi,c}(t) - x) \|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau}},$$

其中  $\phi(\cdot)$  表示标准正态分布的概率密度函数,  $\Delta B_t = B(t + \tau) - B(t)$ . 由定义,可得:

$$\alpha = P\{L^{\pi,c}(t) \leq VaR_\alpha\} = P\{X^{\pi,c}(t)[1 - \exp\{H(t, \pi(t), c(t))\tau + \pi^T(t)\sigma(t)P_q \Delta B_t\}] \leq$$

$$VaR_\alpha\} = P\left\{\frac{X^{\pi,c}(t) - VaR_\alpha}{X^{\pi,c}(t) \exp\{H(t, \pi(t), c(t))\tau\}} \leq \exp\{\pi^T(t)\sigma(t)P_q \Delta B_t\}\right\} =$$

$$P\left\{\frac{1}{\|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau}} \ln \frac{X^{\pi,c}(t) - VaR_\alpha}{X^{\pi,c}(t) \exp\{H(t, \pi(t), c(t))\tau\}} \leq \frac{\pi^T(t)\sigma(t)P_q \Delta B_t}{\|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau}}\right\},$$

则有

$$1 - \alpha = \Phi\left(\frac{1}{\|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau}} \ln \frac{X^{\pi,c}(t) - VaR_\alpha}{X^{\pi,c}(t) \exp\{H(t, \pi(t), c(t))\tau\}}\right),$$

$$\frac{1}{\|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau}} \ln \frac{X^{\pi,c}(t) - VaR_\alpha}{X^{\pi,c}(t) \exp\{H(t, \pi(t), c(t))\tau\}} = \Phi^{-1}(1 - \alpha) \triangleq b_1.$$

因此  $VaR_\alpha = X^{\pi,c}(t) [1 - \exp\{H(t, \pi(t), c(t))\tau + \|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau} b_1\}]$ , 从而  $CVaR_\alpha(L^{\pi,c}(t)) =$

$$E[L^{\pi,c}(t) | L^{\pi,c}(t) \geq VaR_\alpha] = \frac{1}{1 - \alpha} \int_{VaR_\alpha}^{+\infty} x f_L(x) dx. \text{ 在 } x f_L(x) \text{ 的表达式中, 令}$$

$$\frac{1}{\|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau}} \ln \frac{X^{\pi,c}(t) - x}{X^{\pi,c}(t) \exp\{H(t, \pi(t), c(t))\tau\}} = t,$$

则

$$X^{\pi,c}(t) - X^{\pi,c}(t) \exp\{H(t, \pi(t), c(t))\tau + \|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau} t\} = x,$$

$$dx = -X^{\pi,c}(t) [\exp\{H(t, \pi(t), c(t))\tau + \|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau} t\}] \cdot \|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau} dt.$$

因此:

$$\begin{aligned} \text{CVaR}_\alpha(L^{\pi,c}(t)) &= \frac{1}{1-\alpha} \int_{b_1}^{-\infty} -X^{\pi,c}(t) [1 - \exp\{H(t, \pi(t), c(t))\tau + \|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau} t\}] \cdot \\ &\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \frac{X^{\pi,c}(t)}{1-\alpha} \int_{-\infty}^{b_1} [1 - \exp\{H(t, \pi(t), c(t))\tau + \|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau} t\}] \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \\ &\frac{X^{\pi,c}(t)}{1-\alpha} [\Phi(b_1) - \exp\{H(t, \pi(t), c(t))\tau + \frac{1}{2} \|\pi^T(t)\sigma(t)P_q\|^2 \tau\} \cdot \Phi(b_1 - \|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau})] = \\ &X^{\pi,c}(t) [1 - \exp\{H(t, \pi(t), c(t))\tau + \frac{1}{2} \|\pi^T(t)\sigma(t)P_q\|^2 \tau\} \cdot \frac{\Phi(b_1 - \|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau})}{1-\alpha}] = \\ &X^{\pi,c}(t) [1 - b_2 \exp\{H(t, \pi(t), c(t))\tau + \frac{1}{2} \|\pi^T(t)\sigma(t)P_q\|^2 \tau\}], \end{aligned}$$

其中  $b_2 = \frac{\Phi(b_1 - \|\pi^T(t)\sigma(t)P_q\| \sqrt{\tau})}{1-\alpha}$ . 定理 1 得证.

下面将使用相对 CVaR 约束, 即  $\text{CVaR}_\alpha(L^{\pi,c}(t))/X^{\pi,c}(t)$ . 令  $\alpha^* \in (0, 1)$  为相对 CVaR 约束的基准参数, 则相对 CVaR 约束如下:

$$1 - b_2(t) \exp\left\{H(t, \pi(t), c(t))\tau + \frac{1}{2} \|\pi^T(t)\sigma(t)P_q\|^2 \tau\right\} \leq \alpha^*. \quad (5)$$

## 2 HJB 方程与最优投资组合选择

假设  $U_1(x)$  和  $U_2(x)$  分别是消费和最终财富的效用函数, 满足以下条件: (A1)  $U_1(\cdot)$  和  $U_2(\cdot)$  都是二次可微凹函数; (A2)  $U_1(\cdot)$  和  $U_2(\cdot)$  都是递增函数且满足  $\lim_{x \rightarrow +\infty} U_1'(x) = \lim_{x \rightarrow +\infty} U_2'(x) = 0$  和  $\lim_{x \rightarrow 0^+} U_1'(x) = \lim_{x \rightarrow 0^+} U_2'(x) = +\infty$ .

利用经济学的效用理论, 投资者选择一个投资组合策略过程  $\{\pi(t), c(t)\}$ , 期望他们的消费效用和最终财富的价值最大化. 令  $\rho > 0$  是折现因子, 那么期望的折现效用为  $E[\int_0^T e^{-\rho t} U_1(C(t)) dt + e^{-\rho T} U_2(X^{\pi,c}(T))]$ , 且  $X^{\pi,c}(0) = X(0)$ . 由关于最优投资组合和消费策略的动态规划方法, 得最优价值函数为

$$V(x, t) = \sup_{(\pi(t), c(t))} E_t \left[ \int_0^T e^{-\rho t} U_1(C(t)) dt + e^{-\rho T} U_2(X^{\pi,c}(T)) \right],$$

其中  $E_t$  表示条件期望,  $X^{\pi,c}(0) = X(0)$ . 因此, HJB 方程可以表示为

$$\frac{\partial V}{\partial t} + \sup_{(\pi(t), c(t))} \left[ e^{-\rho t} U_1(C(t)) + x(r(t) - c(t) + \pi^T(t)\mu^*(t)) \frac{\partial V}{\partial X} + \frac{\|\pi^T(t)\sigma(t)P_q\|^2 x^2}{2} \cdot \frac{\partial^2 V}{\partial X^2} \right] = 0,$$

这里边界条件  $V(x, T) = e^{-\rho T} U_2(x)$ . 在本文中, 令  $U_1(x) = U_2(x) = \ln(x) \triangleq U(x)$ , 有

$$\ln(X^{\pi,c}(t)) = \ln(X(0)) + \int_0^t H(s, \pi(s), c(s)) ds + \int_0^t \pi^T(s)\sigma(s)P_q dB(s),$$

则消费的效用函数为

$$\begin{aligned} \int_0^T e^{-\rho t} \ln(c(t)) X^{\pi,c}(t) dt &= \int_0^T e^{-\rho t} \ln(c(t)) dt + \frac{1 - e^{-\rho T}}{\rho} \ln X(0) + \\ &\int_0^T e^{-\rho t} dt \int_0^t H(s, \pi(s), c(s)) ds + \int_0^T e^{-\rho t} dt \int_0^t \pi^T(s)\sigma(s)P_q dB(s), \end{aligned}$$

对上式使用 Fubini 定理,可得:

$$\int_0^T e^{-\rho t} dt \int_0^t H(s, \pi(s), c(s)) ds = \int_0^T H(s, \pi(s), c(s)) ds \int_s^T e^{-\rho t} dt = \int_0^T \frac{e^{-\rho s} - e^{-\rho T}}{\rho} H(s, \pi(s), c(s)) ds.$$

因此,

$$\int_0^T e^{-\rho t} \ln(c(t) X^{\pi, c}(t)) dt = \frac{1 - e^{-\rho T}}{\rho} \ln X(0) + \int_0^T e^{-\rho t} [\ln(c(t)) + \frac{1}{\rho} (1 - e^{-\rho(T-t)}) H(t, \pi(t), c(t))] dt + \int_0^T e^{-\rho t} dt \int_0^t \pi^T(s) \sigma(s) P_q dB(s),$$

对上述等式两边都取期望,可得:

$$\begin{aligned} E \left[ \int_0^T e^{-\rho t} \ln(c(t) X^{\pi, c}(t)) dt \right] &= \frac{1 - e^{-\rho T}}{\rho} \ln X(0) + \\ E \left\{ \int_0^T e^{-\rho t} \left[ \ln(c(t)) + \frac{1}{\rho} (1 - e^{-\rho(T-t)}) H(t, \pi(t), c(t)) \right] dt \right\}, \end{aligned}$$

同理可得:

$$E [e^{-\rho t} \ln(X^{\pi, c}(t))] = e^{-\rho t} \ln X(0) + e^{-\rho t} E \left[ \int_0^T H(t, \pi(t), c(t)) dt \right].$$

则由(5)表示的 CVaR 约束下的优化问题可以描述为

$$\begin{cases} \sup_{(\pi(t), c(t))} \left\{ \left[ \frac{1}{\rho} (1 - e^{-\rho t}) + e^{-\rho t} \right] \ln X(0) + E \left[ \int_0^T e^{-\rho t} [\ln(c(t)) + \frac{1}{\rho} (1 - (1 - \rho) e^{-\rho(T-t)}) H(t, \pi(t), c(t))] dt \right] \right\}, \\ \text{s.t. } 1 - b_2(t) \exp\{H(t, \pi(t), c(t))\tau + \frac{1}{2} \|\pi^T(t) \sigma(t) P_q\|^2 \tau\} \leq \alpha^*, \end{cases} \quad (6)$$

易知,优化问题(6)等同于下面的优化问题:

$$\begin{cases} \sup_{(\pi(t), c(t))} [\ln(c(t)) + \frac{1}{\rho} (1 - (1 - \rho) e^{-\rho(T-t)}) H(t, \pi(t), c(t))], \\ \text{s.t. } 1 - b_2(t) \exp\{H(t, \pi(t), c(t))\tau + \frac{1}{2} \|\pi^T(t) \sigma(t) P_q\|^2 \tau\} \leq \alpha^*, \end{cases} \quad (7)$$

下面的定理 2 给出了优化问题(7)的解,定义

$$Q(\pi(t), c(t)) \triangleq 1 - b_2(t) \exp\{H(t, \pi(t), c(t))\tau + \frac{1}{2} \|\pi^T(t) \sigma(t) P_q\|^2 \tau\}.$$

**定理 2** 由(7)式给出的最优化问题的解是

$$\pi^*(t) = \begin{cases} [\sigma(t) P_q P_q^T \sigma^T(t)]^{-1} \mu^*(t), & \text{若 } Q(\pi(t), c(t)) < \alpha^*, \\ \gamma_1 [\sigma(t) P_q P_q^T \sigma^T(t)]^{-1} \mu^*(t), & \text{若 } Q(\pi(t), c(t)) = \alpha^*, \end{cases}$$

和

$$c^*(t) = \begin{cases} \frac{\rho}{1 - (1 - \rho) \exp\{-\rho(T-t)\}}, & \text{若 } Q(\pi(t), c(t)) < \alpha^*, \\ \frac{\gamma_2 \rho}{1 - (1 - \rho) \exp\{-\rho(T-t)\}}, & \text{若 } Q(\pi(t), c(t)) = \alpha^*, \end{cases}$$

其中  $\gamma_1$  为如下关于  $x$  的方程的根.

$$(1 - \alpha^*)(1 - \alpha) = \Phi(b_1 - x \|\pi_M^T(t) \sigma(t) P_q\| \sqrt{\tau}) \exp\{[x \pi_M^T(t) \mu^*(t) + r(t) - f(x) c_M(t)] \tau\}, \quad (8)$$

其中  $\pi_M(t) = (\sigma(t) P_q P_q^T \sigma^T(t))^{-1} \mu^*$ ,  $c_M(t) = \frac{\rho}{1 - (1 - \rho) \exp\{-\rho(T-t)\}}$ ,

$$\gamma_2 = 1 + \frac{[\gamma_1 \|\pi_M^T(t) \sigma(t) P_q\|^2 - \pi_M^T(t) \mu^*(t)]}{\frac{\phi(b_1 - \gamma_1 \|\pi_M^T(t) \sigma(t) P_q\| \sqrt{\tau})}{\Phi(b_1 - \gamma_1 \|\pi_M^T(t) \sigma(t) P_q\| \sqrt{\tau})} \cdot \frac{\|\pi_M^T(t) \sigma(t) P_q\|}{\sqrt{\tau}} - \gamma_1 \|\pi_M^T(t) \sigma(t) P_q\|^2}. \quad (9)$$

**证明** 易知,目标函数是关于  $c(t)$  和  $\pi(t)$  的凹函数,可行域是凸的.定义拉格朗日函数为

$$LA(t, \pi(t), c(t), \lambda(t)) = \ln c(t) + \frac{1}{\rho} [1 - (1 - \rho) \exp\{-\rho(T - t)\} H(t, \pi(t), c(t))] - \lambda(t) [1 - b_2(t) \exp\{H(t, \pi(t), c(t))\tau + \frac{1}{2} \|\pi^\top(t) \sigma(t) P_q\|^2 \tau\} - \alpha^*],$$

其中  $\lambda(t)$  为拉格朗日乘子,设  $(\pi^*, c^*, \lambda^*)$  为拉格朗日函数的 KKT 点,

$$\begin{cases} \frac{\partial}{\partial \pi} LA(t, \pi, c, \lambda) |_{(\pi^*, c^*, \lambda^*)} = 0, \\ \frac{\partial}{\partial c} LA(t, \pi, c, \lambda) |_{(\pi^*, c^*, \lambda^*)} = 0, \\ [1 - b_2(t) \exp\{H(t, \pi(t), c(t))\tau + \frac{1}{2} \|\pi^\top \sigma(t) P_q\|^2 \tau\} - \alpha^*]_{(\pi^*, c^*, \lambda^*)} \leq 0, \lambda^* \geq 0, \\ \{\lambda [1 - b_2(t) \exp\{H(t, \pi(t), c(t))\tau + \frac{1}{2} \|\pi^\top \sigma(t) P_q\|^2 \tau\} - \alpha^*]\}_{(\pi^*, c^*, \lambda^*)} = 0, \end{cases}$$

当  $Q(\pi(t), c(t)) < \alpha^*$  时,(7)式中的最优化问题简化为下面的无约束问题

$$\sup_{(\pi(t), c(t))} \left[ \ln c(t) + \frac{1}{\rho} [1 - (1 - \rho) \exp\{-\rho(T - t)\} H(t, \pi(t), c(t))] \right].$$

由条件,最优解  $(\pi_M(t), c_M(t))$  满足:

$$\begin{cases} \mu^*(t) - \sigma(t) P_q P_q^\top \sigma^\top(t) \pi(t) |_{(\pi_M(t), c_M(t))} = 0, \\ \frac{1}{c(t)} - \frac{1 - (1 - \rho) \exp\{-\rho(T - t)\}}{\rho} |_{(\pi_M(t), c_M(t))} = 0. \end{cases}$$

当  $Q(\pi(t), c(t)) = \alpha^*$  时,令  $\frac{\partial}{\partial \pi} LA(t, \pi, c, \lambda) = 0$ ,可得  $\frac{1 - (1 - \rho) \exp\{-\rho(T - t)\}}{\rho} [\mu^*(t) - \sigma(t) P_q P_q^\top \sigma^\top(t) \pi(t)] +$

$\lambda \left\{ \left[ \frac{\partial b_2(t)}{\partial \pi} + b_2(t) \mu^*(t) \tau \right] \exp\{(\pi(t) \mu^*(t) + r(t) - c(t))\tau\} \right\} = 0$ ,令  $\pi^*(t) = \gamma_1 \pi_M(t)$ ,  $c^*(t) = \gamma_2 c_M(t)$ ,

则一定存在唯一的  $(\gamma_1, \gamma_2)$  为下面问题的最优解

$$\begin{cases} \sup_{(\gamma_1, \gamma_2)} \left[ \ln(\gamma_2 c_M(t)) + \frac{1}{\rho} (1 - \rho) e^{-\rho(T-t)} H(t, \gamma_1 \pi_M(t), \gamma_2 c_M(t)) \right], \\ \text{s.t. } 1 - b_2(t) \exp\{H\tau + \frac{1}{2} \|\gamma_1 \pi_M^\top(t) \sigma(t) P_q\|^2 \tau\} = \alpha^*, \end{cases}$$

由一阶必要条件,必存在一个唯一的拉格朗日乘子  $\lambda^*$ ,使得:

$$\begin{aligned} \nabla \{ \ln(\gamma_2 c_M(t)) + \frac{1}{\rho} (1 - (1 - \rho) e^{-\rho(T-t)}) H(t, \gamma_1 \pi_M(t), \gamma_2 c_M(t)) \} = \\ \lambda^* \nabla \{ 1 - b_2(t) \exp\{H(t, \pi(t), c(t))\tau + \frac{1}{2} \|\gamma_1 \pi_M^\top(t) \sigma(t) P_q\|^2 \tau\} - \alpha^* \}, \end{aligned}$$

通过消除  $\lambda^*$ ,方程(9)成立,并且  $\gamma_1$  由方程(8)给出.定理2得证.

### 3 实证分析

本节中,通过实证分析来展示本文所得的结果.不失一般性,假设  $m=4$ ,即选择中国金融市场的4只股票的日收盘价数据,股票代码分别为000800,601006,601933和601377,时间跨度都是从2013年1月1日到2017年12月31日,数据来自Wind数据库.

表1给出了4只股票收益的统计特征.从表1可以看出,4只股票的峰度系数分别为6.1536,9.9173,159.47和131.58,它们均大于正态分布的峰度系数3.

由4只股票的日收盘数据,可以分别计算出股票的收益和协方差矩阵:

$$\mu = (0.0957, 0.0883, -0.2757, -0.1620)^\top,$$

$$\sigma^T \sigma = \begin{pmatrix} 0.001\ 0 & 0.000\ 3 & 0.000\ 4 & 0.000\ 4 \\ 0.000\ 3 & 0.000\ 5 & 0.000\ 2 & 0.000\ 3 \\ 0.000\ 4 & 0.000\ 2 & 0.001\ 9 & 0.000\ 3 \\ 0.000\ 4 & 0.000\ 3 & 0.000\ 3 & 0.001\ 2 \end{pmatrix}.$$

表 1 4 只股票的基本统计特征

Tab. 1 Basic statistical characteristics of four stocks

stock codes	Mean	Standard deviation	Kurtosis	J-B test	P-Value
000800	0.095 7	0.032 3	6.153 6	0.042 94	0.000 0
601006	0.088 3	0.021 8	9.917 3	0.211 1	0.000 0
601933	-0.275 7	0.043 8	159.47	0.468 0	0.000 0
601377	-0.162 0	0.034 2	131.58	0.924 2	0.016 7

取置信水平  $\alpha = 0.95, \alpha^* = 0.06$ , 非广延参数  $q = 1.5, T = 5(a)$ , 利率  $r = 0.018$ , 另取折现因子  $\rho = 0.018$ . 取范围  $\tau = 0.02$ , 即一周左右.

图 1 为在非广延统计理论下不同时间无风险资产的最优投资组合. 图 2 为在非广延统计理论下不同时间的最优消费率.

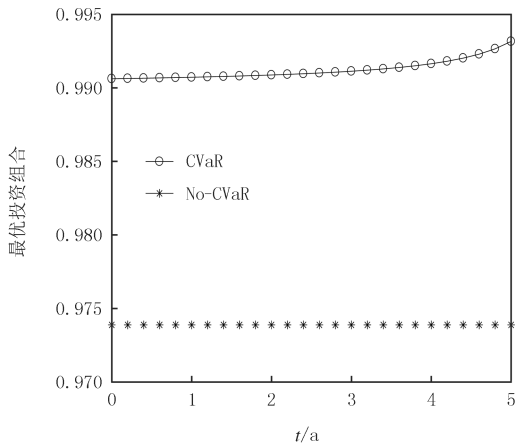


图1 无风险资产的最优投资组合

Fig. 1 Optimal portfolio of risk-free assets

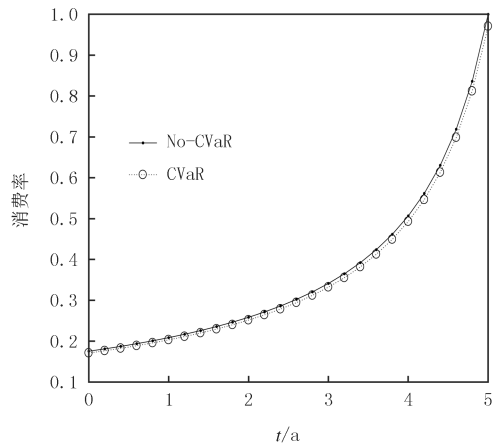


图2 非广延统计理论下的最优消费率

Fig. 2 Optimal consumption rate under non-extensive statistical theory

从图 1 可以看出, 在无 CVaR 约束下, 投资无风险资产的财富占总财富的比例为常数. 在有 CVaR 约束时, 在非广延统计理论下, 投资风险资产的财富占总财富的比例不再是一个常数, 随着时间的变化, 该比例逐渐降低. 由图 2 可以看出, 有 CVaR 约束和无 CVaR 约束的最优消费率相差很小.

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## Optimal portfolio selection under the constraint of CVaR in time-varying coefficient model

——Based on the non-extensive statistical theory

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**Abstract:** Considering that economic variables vary over time, it is reasonable to assume that the returns and covariance matrices of assets are time-varying functions. Moreover, several empirical results show that the distributions of returns for risky assets have appeared the characteristics of fat tails. Therefore, we apply Tsallis distribution to driving the returns of risky assets, which can capture this characteristics of fat tails better. In this paper, we propose continuous-time optimal portfolio selection problem with CVaR constraint under non-extensive statistical mechanics. According to the time-varying coefficient in the model, the local constant fitting method is adopted. We first construct the price process of risky asset under non-extensive statistical mechanics. Then, the HJB equation is obtained by using the stochastic dynamic programming method. Furthermore, we propose the optimal portfolio strategy with CVaR constraint under the logarithmic utility function by Lagrange multiplier techniques. Empirical analysis is discussed to show the fitting effect of the obtained conclusions.

**Keywords:** non-extensive statistical mechanics; portfolio selection; CVaR constraint; time-varying model; HJB equation